COUNTING JOINTS WITH MULTIPLICITIES

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Abstract. Let $\mathfrak{L}$ be a collection of L lines in $\mathbb{R}^3$ and J the set of joints formed by $\mathfrak{L}$, i.e. the set of points each of which lies in at least 3 non-coplanar lines of $\mathfrak{L}$. It is known that $|J| \lesssim L^{3/2}$ (first proved by Guth and Katz). For each joint $x \in J$, let the multiplicity $N(x)$ of x be the number of triples of non-coplanar lines through x. We prove here that $\sum_{x\in J} N(x)^{1/2} \lesssim L^{3/2}$, while in the last section we extend this result to real algebraic curves of bounded degree in $\mathbb{R}^3$, as well as to curves in $\mathbb{R}^3$ parametrised by polynomials of bounded degree.

1. Introduction

A point $x \in \mathbb{R}^n$ is a joint for a collection $\mathfrak{L}$ of lines in $\mathbb{R}^n$ if there exist at least n lines in $\mathfrak{L}$ whose directions span $\mathbb{R}^n$. The problem of bounding the number of joints by a power of the number of the lines forming them first appeared in [CEG$^+$92], where it was proved that if J is the set of joints formed by a collection of L lines in $\mathbb{R}^3$, then $|J| = O(L^{7/4})$. Successive progress was made in improving the upper bound of $|J|$ in three dimensions, by Sharir, Sharir and Welzl, and Feldman and Sharir (see [Sha94], [SW04], [FS05]). Wolff had already observed in [Wol99] that there exists a connection between the joints problem and the Kakeya problem, and, using this fact, Bennett, Carbery and Tao found an improved upper bound for $|J|$, with a particular assumption on the angles between the lines forming each joint (see [BCT06]). Eventually, Guth and Katz provided a sharp upper bound in [GK08]; they showed that, in $\mathbb{R}^3$, $|J| = O(L^{3/2})$. The proof was an adaptation of Dvir's algebraic argument in [Dvi09] for the solution of the finite field Kakeya problem, which involves working with the zero set of a polynomial. Dvir, Guth and Katz induced dramatic developments with this work, because they introduced for the first time algebraic techniques to approach combinatorial problems. Further work was done by Elekes, Kaplan and Sharir in [EKS11], and finally, a little later, Kaplan, Sharir and Shustin (in [KSS10]) and Quilodrán (in [Qui10]) independently solved the joints problem in n dimensions, using again algebraic techniques, simpler than in [GK08].

In particular, Quilodrán and Kaplan, Sharir and Shustin showed that, if $\mathfrak{L}$ is a collection of L lines in $\mathbb{R}^n$, $n \geq 2$, and J is the set of joints formed by $\mathfrak{L}$, then

$$|J| \leq c_n \cdot L^{\frac{n}{n-1}}, \tag{1}$$

where c_n is a constant depending only on the dimension n.

In this setting, we define the multiplicity $N(x)$ of a joint x as the number of n-tuples of lines of $\mathfrak{L}$ through x, whose directions span $\mathbb{R}^n$; we mention here that we consider the n-tuples to be unordered, although considering them ordered would not cause any substantial change in what follows.

From (1) we know that $\sum_{x\in J} 1 \leq c_n \cdot L^{\frac{n}{n-1}}$. A question by Anthony Carbery is if one can improve this to get

$$\sum_{x\in J} N(x)^{\frac{1}{n-1}} \leq c_n' \cdot L^{\frac{n}{n-1}}, \tag{2}$$

where c_n' is, again, a constant depending only on n. We clarify here that the choice of $\frac{1}{n-1}$ as the power of the multiplicities $N(x)$ on the left-hand side of (2) does not affect the truth of (2) when each joint has multiplicity 1, while it is the largest power of $N(x)$ that one can hope for, since it is the largest power of $N(x)$ that makes (2) true when all the lines of $\mathfrak{L}$ are passing through the same point and each n of them are linearly independent (in which case the point is a joint of multiplicity $\binom{L}{n} \sim L^n$). Also, (2) obviously holds when $n = 2$: in that case, the left-hand side is smaller than the number of all the 2-tuples of the L lines, i.e. than $\binom{L}{2} \sim L^2$.

In fact, the above question can also be seen from a harmonic analytic point of view (again, see [Wol99]). Specifically, if T_ω, for $\omega \in \Omega \subset S^{n-1}$, are tubes in $\mathbb{R}^n$ with eccentricity $\frac{1}{\delta}$ such that their directions $\omega \in \Omega$ are δ-separated, then the Kakeya maximal operator conjecture asks for a sharp upper bound of the quantity

$$\int_{x\in\mathbb{R}^n} \big(\sum_{\omega\in\Omega} \chi_{T_\omega}(x)\big)^{\frac{n}{n-1}} dx = \int_{x\in\mathbb{R}^n} \#\{\text{tubes } T_\omega \text{ through } x\}^{\frac{n}{n-1}} dx.$$

On the other hand, in the case when a collection $\mathfrak{L}$ of lines in $\mathbb{R}^n$ has the property that, whenever n of the lines meet at a point, they form a joint there, then, for all $x \in J$, $N(x) \simeq \#\{\text{lines of } \mathfrak{L} \text{ through } x\}^n$, and thus the left-hand side of (2) is

$$\simeq \sum_{x\in J} \#\{\text{lines of } \mathfrak{L} \text{ through } x\}^{\frac{n}{n-1}}.$$

Therefore, in both cases, the problem lies in bounding analogous quantities.

We will indeed show here that (2) holds in $\mathbb{R}^3$:

Theorem 1.1. *Let $\mathfrak{L}$ be a collection of L lines in $\mathbb{R}^3$, forming a set of joints J. Then,*

$$\sum_{x\in J} N(x)^{1/2} \leq c \cdot L^{3/2},$$

where c is a constant independent of $\mathfrak{L}$.

The basic tool for the proof of Theorem 1.1 will be the Guth - Katz polynomial method, developed by Guth and Katz in [GK10].

More particularly, we first need to consider, for all $N \in \mathbb{N}$, the subset J_N of J, defined as follows:

$J_N := \{x \in J : N \leq N(x) < 2N\}$.

In addition, we define, for all $N, k \in \mathbb{N}$, the following subset of J_N:

$J_N^k := \{x \in J_N$: x intersects at least k and less than $2k$ lines of $\mathfrak{L}\}$.

We will then apply the Guth-Katz polynomial method in the same way as in [GK10], from which we will decuce that:

Proposition 1.2. *If $\mathfrak{L}$ is a collection of L lines in $\mathbb{R}^3$ and $N, k \in \mathbb{N}$, then*

$$|J_N^k| \cdot N^{1/2} \leq c \cdot \left(\frac{L^{3/2}}{k^{1/2}} + \frac{L}{k} \cdot N^{1/2} \right),$$

where c is a constant independent of $\mathfrak{L}$, N and k.

Theorem 1.1 will then follow from Proposition 1.2 (details are displayed at the end of section 3). Finally, in the somewhat independent section 4 of the paper we generalise the statement of Theorem 1.1 for joints formed by real algebraic curves in $\mathbb{R}^3$ of bounded degree, as well as curves in $\mathbb{R}^3$ parametrised by real polynomials of bounded degree (Theorem 4.2.1).

Note that the proof of (2) for three dimensions that we are providing cannot be applied for higher dimensions, as a crucial part of it is that the number of critical lines of an algebraic surface in $\mathbb{R}^3$ is bounded (details in section 2.2), a fact which is not always true in higher dimensions.

We clarify here that, in whatever follows, any expression of the form $A \lesssim B$ means that there exists an explicit, non-negative constant M, depending only on the dimension, such that $A \leq M \cdot B$. And now, before continuing with the proof, we present certain facts and tools which will prove useful to our goal.

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2. Background

2.1. **The Guth - Katz polynomial method.** As we have already mentioned, this technique will be the basic tool for our proof. The method can be applied in $\mathbb{R}^n$, for all $n \in \mathbb{N}$, and results in a decomposition of $\mathbb{R}^n$ by the

zero set of a polynomial. All the details are fully explained in [GK10], but we are presenting here the basic result and the theorems leading to it.

We start with a result of Stone and Tukey, known as the polynomial ham sandwich theorem (that is, in fact, a consequence of the Borsuk - Ulam theorem). In particular, we say that the zero set of a polynomial $p \in \mathbb{R}[x_1, ..., x_n]$ bisects a set $U \subset \mathbb{R}^n$ of finite, positive volume, when the sets $U \cap \{p > 0\}$ and $U \cap \{p < 0\}$ have the same volume.

Theorem 2.1.1. (Stone, Tukey, [ST42]) *Let U_1, ..., U_M be sets in $\mathbb{R}^n$ of finite, positive volume, where $M = \binom{n+d}{n}$. Then, there exists a polynomial of degree $\leq d$, whose zero set bisects each U_i.*

In analogy to this, if S is a finite set of points in $\mathbb{R}^n$, we say that the zero set of a polynomial $p \in \mathbb{R}[x_1, ..., x_n]$ bisects S if the sets $S \cap \{p > 0\}$ and $S \cap \{p < 0\}$ each contain at most half of the points of S. Now, Guth and Katz, using the above theorem, proved the following:

Corollary 2.1.2. (Guth, Katz, [GK10, Corollary 4.4] *Let S_1, ..., S_M be disjoint, finite sets of points in $\mathbb{R}^n$, where $M = \binom{n+d}{n}$. Then, there exists a polynomial of degree $\leq d$, whose zero set bisects each S_i.*

Another proof of this appears in [KMS11], using [Mat03].

The Guth - Katz polynomial method consists of successive applications of this last corollary. We now state the result of the application of the method, while its proof is the method itsellf.

Theorem 2.1.3. (Guth, Katz, [GK10, Theorem 4.1]) *Let $\mathfrak{G}$ be a finite set of S points in $\mathbb{R}^n$, and $d > 1$. Then, there exists a polynomial $p \in \mathbb{R}[x_1, ..., x_n]$ of degree $\leq d$, whose zero set decomposes $\mathbb{R}^n$ in $\simeq d^n$ cells, each of which contains $\lesssim S/d^n$ points of $\mathfrak{G}$.*

Proof. We find polynomials p_1, ..., $p_J \in \mathbb{R}[x_1, ..., x_n]$, in the following way:

By Corollary 2.1.2 applied to the finite set of points $\mathfrak{G}$, there exists a polynomial $p_1 \in \mathbb{R}[x_1, ..., x_n]$, of degree $\lesssim 1^{1/n}$, whose zero set Z_1 bisects $\mathfrak{G}$. Thus, $\mathbb{R}^n \setminus Z_1$ consists of 2^1 disjoint cells (the cell $\{p_1 > 0\}$ and the cell $\{p_1 < 0\}$), each of which contains $\lesssim S/2^1$ points of $\mathfrak{G}$.

By Corollary 2.1.2 applied to the disjoint, finite sets of points $\mathfrak{G} \cap \{p_1 > 0\}$, $\mathfrak{G} \cap \{p_1 < 0\}$, there exists a polynomial $p_2 \in \mathbb{R}[x_1, ..., x_n]$, of degree $\lesssim 2^{1/n}$, whose zero set Z_2 bisects $\mathfrak{G} \cap \{p_1 > 0\}$ and $\mathfrak{G} \cap \{p_1 < 0\}$. Thus, $\mathbb{R}^n \setminus (Z_1 \cup Z_2)$ consists of 2^2 disjoint cells (the cells $\{p_1 > 0\} \cap \{p_2 > 0\}, \{p_1 > 0\} \cap \{p_2 < 0\}, \{p_1 < 0\} \cap \{p_2 > 0\}$ and $\{p_1 < 0\} \cap \{p_2 < 0\}$), each of which contains $\lesssim S/2^2$ points of $\mathfrak{G}$.

We continue in a similar way; by the end of the j-th step, we have produced polynomials p_1, ..., p_j, with degrees $\lesssim 2^{(1-1)/n}$, ..., $\lesssim 2^{(j-1)/n}$ respectively, such that $\mathbb{R}^n \setminus (Z_1 \cup ... \cup Z_j)$ consists of 2^j disjoint cells, each of which contains $\lesssim S/2^j$ points of $\mathfrak{G}$.

We stop this procedure at the J-th step, where J is such that the polynomial $p := p_1 \cdots p_J$ has degree $\leq d$ and the number of cells in which

$\mathbb{R}^n \setminus (Z_1 \cup ... \cup Z_J)$ is decomposed is $\simeq d^n$ (in other words, we stop when $2^{(1-1)/n} + 2^{(2-1)/n} + ... + 2^{(J-1)/n} \leq d$ and $2^J \simeq d^n$). The polynomial p has the properties that we want (note that its zero set is the set $Z_1 \cup ... \cup Z_J$).

□

2.2. **More preliminaries.** The great advantage of applying the Guth - Katz polynomial method for decomposing $\mathbb{R}^3$ and, at the same time, a finite set of points $\mathfrak{G}$ in $\mathbb{R}^3$, does not only lie in the fact that it allows us to have a control over the number of points of $\mathfrak{G}$ in the interior of each cell; it lies in the fact that the surface that partitions $\mathbb{R}^3$ is the zero set of a polynomial. This immediately gives us a control over many quantities. In particular:

Theorem 2.2.1. (Guth, Katz, [GK08, Corollary 2.5]) *(Corollary of Bézout's Theorem) Let p_1, $p_2 \in \mathbb{R}[x, y, z]$. If p_1, p_2 do not have a common factor, then there exist at most $\deg p_1 \cdot \deg p_2$ lines simultaneously contained in the zero set of p_1 and the zero set of p_2.*

An application of this result enables us to bound the number of critical lines of an algebraic surface of degree d in $\mathbb{R}^3$. More particularly:

Definition. *Let Z be the zero set of a polynomial $p \in \mathbb{R}[x, y, z]$. A critical point x of Z is a point for which $p(x) = 0$ and $\nabla p(x) = 0$. Any other point of Z is called a regular point. A line contained in Z is called a critical line if each point of the line is critical.*

Note that if x is a regular point of the zero set Z of a polynomial $p \in \mathbb{R}[x, y, z]$, then, by the implicit function theorem, Z is locally a manifold in a neighbourhood of x and the tangent space to Z at x is well-defined.

Eventually, the following holds:

Proposition 2.2.2. (Guth, Katz, [GK08, Proposition 3.1]) *Let Z be the zero set of a polynomial $p \in \mathbb{R}[x, y, z]$, of degree $\leq d$. Then, Z contains at most d^2 critical lines.*

And finally:

Definition. *Let $\mathcal{P}$ be a collection of points and $\mathfrak{L}$ a collection of lines in $\mathbb{R}^n$. We say that the pair (p, l), where $p \in \mathcal{P}$ and $l \in \mathfrak{L}$, is an incidence between $\mathcal{P}$ and $\mathfrak{L}$, if $p \in l$. We denote by $I_{\mathcal{P},\mathfrak{L}}$ the number of all the incidences between $\mathcal{P}$ and $\mathfrak{L}$.*

Theorem 2.2.3. (Szemerédi, Trotter, [ST83]) *Let $\mathfrak{L}$ be a collection of L lines in $\mathbb{R}^2$ and $\mathcal{P}$ a collection of P points in $\mathbb{R}^2$. Then,*

$$I_{\mathcal{P},\mathfrak{L}} \leq C \cdot (|P|^{2/3}|L|^{2/3} + |P| + |L|),$$

where C is a constant independent of $\mathfrak{L}$ and $\mathcal{P}$.

This theorem first appeared in [ST83]; other, less complicated proofs have appeared since (see [Szé97] and [KMS11]; in fact, in [KMS11] the proof is a consequence of the Guth - Katz polynomial method).

An immediate consequence of the Szemerédi - Trotter theorem is the following:

Corollary 2.2.4. (Szemerédi, Trotter, [ST83]) *Let $\mathfrak{L}$ be a collection of L lines in $\mathbb{R}^2$ and $\mathfrak{G}$ a collection of S points in $\mathbb{R}^2$, such that each of them intersects at least k lines of $\mathfrak{L}$. Then,*

$$S \leq c_0 \cdot (L^2 k^{-3} + L k^{-1}),$$

where c_0 is a constant independent of $\mathfrak{L}$ and $\mathfrak{G}$.

Note that Corollary 2.2.4 holds not only in $\mathbb{R}^2$, but in $\mathbb{R}^n$ as well, for all $n \in \mathbb{N}$, $n > 2$, by projecting $\mathbb{R}^n$ on a generic plane.

We now continue with the proof of Theorem 1.1 (via the proof of Proposition 1.2).

3. Proof

We start by making certain observations:

Lemma 3.1. *Let x be a joint of multiplicity N for a collection $\mathfrak{L}$ of lines in $\mathbb{R}^3$, such that x lies in $\leq 2k$ of the lines. If, in addition, x is a joint of multiplicity $\leq \frac{N}{2}$ for a subcollection $\mathfrak{L}'$ of the lines, or if it is not a joint at all for the subcollection $\mathfrak{L}'$, then there exist $\geq \frac{N}{1000 \cdot k^2}$ lines of $\mathfrak{L} \setminus \mathfrak{L}'$ passing through x.*

Proof. Since the joint x lies in $\leq 2k$ lines of $\mathfrak{L}$, its multiplicity N is $\leq \binom{2k}{3} \leq 8k^3$. Now:

Let A be the number of lines of $\mathfrak{L} \setminus \mathfrak{L}'$ that are passing through x. We will show that $A \geq \frac{N}{1000 \cdot k^2}$. Indeed, suppose that $A \lneq \frac{N}{1000 \cdot k^2}$. Then,

$$N = \Big|\Big\{\{l_1, l_2, l_3\} : \text{the directions of } l_1,\, l_2,\, l_3 \text{ span } \mathbb{R}^3\Big\}\Big| =$$

$$= \Big|\Big\{\{l_1, l_2, l_3\} : l_1,\, l_2,\, l_3 \in \mathfrak{L}' \text{ and their directions span } \mathbb{R}^3\Big\}\Big| +$$

$$+ \Big|\Big\{\{l_1, l_2, l_3\} : l_1,\, l_2,\, l_3 \in \mathfrak{L} \setminus \mathfrak{L}' \text{ and their directions span } \mathbb{R}^3\Big\}\Big| +$$

$$+ \Big|\Big\{\{l_1, l_2, l_3\} : l_1,\, l_2 \in \mathfrak{L}',\, l_3 \in \mathfrak{L} \setminus \mathfrak{L}' \text{ and their directions span } \mathbb{R}^3\Big\}\Big| +$$

$$+ \Big|\Big\{\{l_1, l_2, l_3\} : l_1,\, l_2 \in \mathfrak{L} \setminus \mathfrak{L}',\, l_3 \in \mathfrak{L}' \text{ and their directions span } \mathbb{R}^3\Big\}\Big| \leq$$

$$\leq \tfrac{N}{2} + \tbinom{A}{3} + \tbinom{A}{2} \cdot 2k + \tbinom{2k}{2} \cdot A \leq$$

$$\leq \tfrac{N}{2} + A^3 + A^2 \cdot 2k + (2k)^2 \cdot A \leq$$

$$\leq \tfrac{N}{2} + \big(\tfrac{N}{1000 \cdot k^2}\big)^3 + \big(\tfrac{N}{1000 \cdot k^2}\big)^2 \cdot 2k + (2k)^2 \cdot \tfrac{N}{1000 \cdot k^2} \leq$$

$\leq \frac{N}{2} + \frac{N}{8} + \frac{N}{8} + \frac{N}{8} \lneq N$ (what we use here is the fact that $N \leq 8k^3$). So, we are led to a contradiction, which means that $A \geq \frac{N}{1000 \cdot k^2}$.

□

Lemma 3.2. *Let x be a joint of multiplicity N for a collection $\mathfrak{L}$ of lines in $\mathbb{R}^3$, such that x lies in $\leq 2k$ of the lines. Then, for every plane containing x, there exist $\geq \frac{N}{1000 \cdot k^2}$ lines of $\mathfrak{L}$ passing through x, which are not lying in the plane.*

Proof. Let $\mathfrak{L}'$ be the set of lines in $\mathfrak{L}$ passing through x and lying in the plane. From Lemma 3.1, we know that there exist $\geq \frac{N}{1000 \cdot k^2}$ lines of $\mathfrak{L} \setminus \mathfrak{L}'$ passing through x, and, by the definition of $\mathfrak{L}'$, these lines do not lie in the plane. Therefore, there exist, indeed, at least $\frac{N}{1000 \cdot k^2}$ lines of $\mathfrak{L}$ passing through x and lying outside the plane.

□

We now continue with the proof of Proposition 1.2. We mention here that our argument will be based to a large extent on the proof of [GK10, Theorem 4.7]. The following presentation, though, is self - contained, and it will be made clear whenever the techniques of [GK10] are repeated.

Proposition 1.2: *If $\mathfrak{L}$ is a collection of L lines in $\mathbb{R}^3$ and $N, k \in \mathbb{N}$, then*

$$|J_N^k| \cdot N^{1/2} \leq c \cdot \left(\frac{L^{3/2}}{k^{1/2}} + \frac{L}{k} \cdot N^{1/2} \right),$$

where c is a constant independent of $\mathfrak{L}$, N and k.

Proof. The proof will be done by induction on the number of lines. Indeed, let $L \in \mathbb{N}$. For c a (non-negative) constant which will be specified later:

- For any collection of lines that consists of 1 line,

$$|J_N^k| \cdot N^{1/2} \leq c \cdot \left(\frac{1^{3/2}}{k^{1/2}} + \frac{1}{k} \cdot N^{1/2} \right), \ \forall \ N, k \in \mathbb{N}$$

(this is obvious, in fact, for any $c \geq 0$, as in this case $J_N = \emptyset$, $\forall \ N \in \mathbb{N}$).

- We assume that

$$|J_N^k| \cdot N^{1/2} \leq c \cdot \left(\frac{L'^{3/2}}{k^{1/2}} + \frac{L'}{k} \cdot N^{1/2} \right), \ \forall \ N, k \in \mathbb{N},$$

for any collection of L' lines, for any $L' \lneq L$.

- We will now prove that

$$|J_N^k| \cdot N^{1/2} \leq c \cdot \left(\frac{L^{3/2}}{k^{1/2}} + \frac{L}{k} \cdot N^{1/2} \right), \tag{3}$$

$\forall \ N, k \in \mathbb{N}$, for any collection of L lines.

We emphasise here that this last claim will be proved for the same constant c as the one appearing in the first two steps of the induction process, provided that this constant is chosen sufficiently large.

Indeed, let $\mathfrak{L}$ be a collection of L lines in $\mathbb{R}^3$ and fix N and k in $\mathbb{N}$. Also, for simplicity, let

$$\mathfrak{G} := J_N^k$$

and

$$S := |J_N^k|$$

for this collection of lines, while $\mathfrak{L}'$ denotes the set of lines in $\mathfrak{L}$ each of which contains $\geq \frac{1}{100}SkL^{-1}$ points of $\mathfrak{G}$. Finally, we mention once again that, in whatever follows, any expression of the form $A \lesssim B$ means that there exists some explicit, non-negative constant M, independent of $\mathfrak{L}$, N and k, such that $A \leq M \cdot B$.

We now proceed in exactly the same way as in the proof of [GK10, Theorem 4.7]:

Each point of $\mathfrak{G}$ has at least k lines of $\mathfrak{L}$ passing through it, so, from the Szemerédi-Trotter theorem, $S \leq c_0 \cdot (L^2k^{-3} + Lk^{-1})$, where c_0 is a constant independent of $\mathfrak{L}$, N and k. Therefore:

If $\frac{S}{2} \leq c_0 \cdot Lk^{-1}$, then $S \cdot N^{1/2} \leq 2c_0 \cdot \frac{L}{k} \cdot N^{1/2}$ (where $2c_0$ is independent of $\mathfrak{L}$, N and k).

Otherwise, $\frac{S}{2} > c_0 \cdot Lk^{-1}$, so, from the Szemeredi-Trotter theorem, $\frac{S}{2} < c_0 \cdot L^2k^{-3}$, which gives $S < 2c_0 \cdot L^2k^{-3}$.

Therefore, the quantity $d := AL^2S^{-1}k^{-3}$ is equal to at least 1 whenever $A \geq 2c_0$; we thus choose A to be large enough for this to hold, and we will specify its value later. Now, using the Guth - Katz polynomial method, we find a polynomial p of degree $\leq d$, whose zero set Z decomposes $\mathbb{R}^3$ in $\lesssim d^3$ cells, each of which contains $\lesssim Sd^{-3}$ points of $\mathfrak{G}$. If there are $\geq 10^{-8}S$ points of $\mathfrak{G}$ in the union of the interiors of the cells, we are in the cellular case. Otherwise, we are in the algebraic case.

Cellular case: We follow the proof of [GK10, Lemma 4.8], to fix A and deduce that $S \cdot N^{1/2} \lesssim L^{3/2}k^{-1/2}$. More particularly:

There are $\gtrsim S$ points of $\mathfrak{G}$ in the union of the interiors of the cells. However, we also know that there exist $\lesssim d^3$ cells in total, each with $\lesssim Sd^{-3}$ points of $\mathfrak{G}$. Therefore, there exist $\gtrsim d^3$ cells, with $\gtrsim Sd^{-3}$ points of $\mathfrak{G}$ in the interior of each. We call the cells with this property "full cells". Now:

• If the interior of some full cell contains $\leq k$ points of $\mathfrak{G}$, then $Sd^{-3} \lesssim k$, so $S \lesssim L^{3/2}k^{-2}$, and since $N \lesssim k^3$, we have that $S \cdot N^{1/2} \lesssim L^{3/2}k^{-1/2}$.

• If the interior of each full cell contains $\geq k$ points of $\mathfrak{G}$, then we will be led to a contradiction by choosing A so large, that there will be too many intersections between the zero set Z of p and the lines of $\mathfrak{L}$ which do not lie in Z. Indeed:

Let $\mathfrak{L}_Z$ be the set of lines of $\mathfrak{L}$ which are lying in Z. Consider a full cell and let S_{cell} be the number of points of $\mathfrak{G}$ in the interior of the cell, $\mathfrak{L}_{cell}$ the set of lines of $\mathfrak{L}$ that intersect the interior of the cell and L_{cell} the number of these lines. Obviously, $\mathfrak{L}_{cell} \subset \mathfrak{L} \setminus \mathfrak{L}_Z$.

Now, each point of $\mathfrak{G}$ has at least k lines of $\mathfrak{L}$ passing through it, therefore each point of $\mathfrak{G}$ lying in the interior of the cell has at least k lines of $\mathfrak{L}_{cell}$ passing through it. Thus, since $S_{cell} \geq k$, we get that $L_{cell} \geq k + (k-1) + (k-2) + ... + 1 \gtrsim k^2$, so

$$L_{cell}^2 k^{-3} \gtrsim L_{cell} k^{-1}.$$

But from the Szemerédi - Trotter theorem,

$$S_{cell} \lesssim L_{cell}^2 k^{-3} + L_{cell} k^{-1}.$$

Therefore, $S_{cell} \lesssim L_{cell}^2 k^{-3}$, so, since we are working in a full cell, $Sd^{-3} \lesssim L_{cell}^2 k^{-3}$, and rearranging we see that

$$L_{cell} \gtrsim S^{1/2} d^{-3/2} k^{3/2}.$$

But each of the lines of $\mathfrak{L}_{cell}$ intersects the boundary of the cell at at least one point, with the property that the induced topology from $\mathbb{R}^3$ to the intersection of the line with the closure of the cell contains an open neighbourhood of x; therefore, there are $\gtrsim S^{1/2} d^{-3/2} k^{3/2}$ incidences of this form between $\mathfrak{L}_{cell}$ and the boundary of the cell. Also, the union of the boundaries of all the cells is the zero set Z of p, and if x is a point of Z which belongs to a line intersecting the interior of a cell, such that the induced topology from $\mathbb{R}^3$ to the intersection of the line with the closure of the cell contains an open neighbourhood of x, then there exists at most one other cell whose interior is also intersected by the line and whose boundary contains x, such that the induced topology from $\mathbb{R}^3$ to the intersection of the line with the closure of that cell contains an open neighbourhood of x. So, if I is the number of incidences between Z and $\mathfrak{L} \setminus \mathfrak{L}_Z$, I_{cell} is the number of incidences between $\mathfrak{L}_{cell}$ and the boundary of the cell, and $\mathcal{C}$ is the set of all the full cells (which, in our case, has cardinality $\gtrsim d^3$), then this last observation leads us to:

$$I \gtrsim \sum_{cell \in \mathcal{C}} I_{cell} \gtrsim (S^{1/2} d^{-3/2} k^{3/2}) \cdot d^3 = S^{1/2} d^{3/2} k^{3/2}.$$

On the other hand, if a line does not lie in the zero set Z of p, then it intersects Z in $\leq d$ points. Thus,

$$I \leq L \cdot d$$

This means that

$$S^{1/2} d^{3/2} k^{3/2} \lesssim L \cdot d,$$

which in turn gives: $A \lesssim 1$. In other words, there exists some constant C, independent of $\mathfrak{L}$, N and k, such that $A \leq C$. By fixing A to be a number larger than C (and of course $\geq 2c_0$, so that $d > 1$), we have a contradiction.

Therefore, in the cellular case there exists some constant c_1, independent of $\mathfrak{L}$, N and k, such that

$$S \cdot N^{1/2} \leq c_1 \cdot \frac{L^{3/2}}{k^{1/2}}.$$

Algebraic case: Let $\mathfrak{G}_1$ denote the set of points in $\mathfrak{G}$ which lie in Z. Here, $|\mathfrak{G}_1| > (1-10^{-8})S$. We analyse the situation:

Since each point of $\mathfrak{G}_1$ intersects at least k lines of $\mathfrak{L}$,

$$I_{\mathfrak{G}_1,\mathfrak{L}} > (1-10^{-8})Sk.$$

And $I_{\mathfrak{G}_1,\mathfrak{L}\setminus\mathfrak{L}'} \leq I_{\mathfrak{G},\mathfrak{L}\setminus\mathfrak{L}'} \leq |\mathfrak{L}\setminus\mathfrak{L}'| \cdot \frac{SkL^{-1}}{100} \leq \frac{1}{100}Sk$.

So, as $I_{\mathfrak{G}_1,\mathfrak{L}'} + I_{\mathfrak{G}_1,\mathfrak{L}\setminus\mathfrak{L}'} = I_{\mathfrak{G}_1,\mathfrak{L}}$, we deduce that

$$I_{\mathfrak{G}_1,\mathfrak{L}'} > (1-10^{-8}-10^{-2})Sk.$$

Thus, there are $\gtrsim Sk$ incidences between $\mathfrak{G}_1$ and $\mathfrak{L}'$; this, combined with the fact that there exist $\leq S$ points of $\mathfrak{G}$ in total, each intersecting $\leq 2k$ lines of $\mathfrak{L}$, allows us to deduce that there exist $\gtrsim S$ points of $\mathfrak{G}_1$, each intersecting $\gtrsim k$ lines of $\mathfrak{L}'$.

Let us now take a moment to look for a practical meaning of this: $\sim S$ of our initial points each lie in $\sim k$ lines of $\mathfrak{L}'$, which is a subset of our initial set of lines $\mathfrak{L}$. Thus, if $\mathfrak{L}'$ is a strict subset of $\mathfrak{L}$, and if many of these points are joints for $\mathfrak{L}'$ with multiplicity $\sim N$, then we can use our induction hypothesis for $\mathfrak{L}'$ and solve the problem if $|\mathfrak{L}'|$ is significantly smaller than L; however, before being able to tackle the problem in the rest of the cases, we need to extract more information.

To that end, we will need to use appropriate, explicit constants now hiding behing the $\gtrsim$ symbols, which we therefore go ahead and find.

More particularly, let $\mathfrak{G}'$ be the set of points of $\mathfrak{G}_1$ each of which intersects $\geq \frac{1-10^{-8}-10^{-2}}{2}k$ lines of $\mathfrak{L}'$.

Since $I_{\mathfrak{G}',\mathfrak{L}'} = I_{\mathfrak{G}_1,\mathfrak{L}'} - I_{\mathfrak{G}_1\setminus\mathfrak{G}',\mathfrak{L}'}$

and $I_{\mathfrak{G}_1\setminus\mathfrak{G}',\mathfrak{L}'} \leq |\mathfrak{G}_1\setminus\mathfrak{G}'| \cdot \frac{1-10^{-8}-10^{-2}}{2}k \leq \frac{1-10^{-8}-10^{-2}}{2}Sk$, we get that

$$I_{\mathfrak{G}',\mathfrak{L}'} > \frac{1-10^{-8}-10^{-2}}{2}Sk.$$

And obviously, $I_{\mathfrak{G}',\mathfrak{L}'} \leq |\mathfrak{G}'| \cdot 2k$. Thus, $\frac{1-10^{-8}-10^{-2}}{2}Sk < |\mathfrak{G}'| \cdot 2k$, and so:

$$|\mathfrak{G}'| \geq \frac{1-10^{-8}-10^{-2}}{4}S;$$

in other words, there exist at least $\frac{1-10^{-8}-10^{-2}}{4}S$ points of $\mathfrak{G}_1$, each intersecting $\geq \frac{1-10^{-8}-10^{-2}}{2}k$ lines of $\mathfrak{L}'$.

But each point of $\mathfrak{G}_1$ lies in Z, so it is either a regular of a critical point of Z. We are thus in one of the following two subcases:

The regular subcase: At least $\frac{10^{-8}}{4}S$ points of $\mathfrak{G}_1$ are regular points of Z.

What we actually need to continue is that Z contains $\gtrsim S$ points of $\mathfrak{G}$ that are regular. Now, if $x \in \mathfrak{G}$ is a regular point of Z, there exists a plane through it, containing all those lines through the point that are lying in Z (otherwise, the point would be a critical point of Z). And, since x is a joint for $\mathfrak{L}$, of multiplicity $\geq N$, lying in $\leq 2k$ lines of $\mathfrak{L}$, by Lemma 3.2 there exist $\gtrsim \frac{N}{k^2}$ lines of $\mathfrak{L}$ passing through x, which are not lying on the plane; so, these lines are not lying in Z, therefore each of them contains $\leq d$ points of $\mathfrak{G}_1$. So, the number of incidences between $\mathfrak{G}_1$ and $\mathfrak{L} \setminus \mathfrak{L}_Z$ is $\gtrsim S \cdot \frac{N}{k^2}$, but also $\leq |\mathfrak{L} \setminus \mathfrak{L}_Z| \cdot d \leq L \cdot d$. Thus: $S \cdot \frac{N}{k^2} \lesssim L \cdot d$, from which we deduce that $S \cdot N^{1/2} \lesssim L^{3/2}k^{-1/2}$.

Therefore,

$$S \cdot N^{1/2} \leq c_2 \cdot \frac{L^{3/2}}{k^{1/2}},$$

for some constant c_2 independent of $\mathfrak{L}$, N and k.

The critical subcase: At most $\frac{10^{-8}}{4}S$ of $\mathfrak{G}_1$ are regular points of Z. Now, either $|\mathfrak{L}'| \geq \frac{L}{100}$ or $|\mathfrak{L}'| < \frac{L}{100}$.

• **If $|\mathfrak{L}'| \geq \frac{L}{100}$:**

(The basic arguments for the proof of this case appear in the proof of [GK10, Proposition 4.7].)

We remember that we are in the algebraic case; thus, $|\mathfrak{G} \setminus \mathfrak{G}_1| < 10^{-8}S$.

Now, let $\mathfrak{L}_1$ be the set of lines in $\mathfrak{L}$, each of which contains $\geq \frac{1}{200}SkL^{-1}$ points of $\mathfrak{G}_1$.

We notice that, if $\frac{1}{400}SkL^{-1} \leq d$, then, by rearranging, we get that $S \lesssim L^{3/2}k^{-2}$, so $S \cdot N^{1/2} \lesssim L^{3/2}k^{-1/2}$ (as $N \lesssim k^3$).

So, from now on we assume that $\frac{1}{400}SkL^{-1} \geq d+1$. Then, each line of $\mathfrak{L}_1$ contains at least $d+1$ points of the zero set Z of p (as $\mathfrak{G}_1$ lies in Z), so each line of $\mathfrak{L}_1$ lies in Z (the degree of p is $\leq d$).

Now, using our assumption that there exist $\geq \frac{L}{100}$ lines in $\mathfrak{L}$, each containing $\geq \frac{1}{100}SkL^{-1}$ points of $\mathfrak{G}$, we will show that $\mathfrak{L}_1$ contains many lines.

Indeed, suppose that $|\mathfrak{L}_1| < \frac{L}{200}$. Then:

We know that $|\mathfrak{G} \setminus \mathfrak{G}_1| < 10^{-8}S$. So, since each point of $\mathfrak{G}$ intersects $\leq 2k$ lines of $\mathfrak{L}$, the points of $\mathfrak{G} \setminus \mathfrak{G}_1$ contribute $\leq 2 \cdot 10^{-8}Sk$ incidences with $\mathfrak{L}$. However, there exist $\geq \frac{L}{100}$ lines in $\mathfrak{L}$, each containing $\geq \frac{1}{100}SkL^{-1}$ points of $\mathfrak{G}$, and since $|\mathfrak{L}_1| < \frac{L}{200}$, at most $\frac{L}{200}$ of them contain $\geq \frac{1}{200}SkL^{-1}$ points of $\mathfrak{G}_1$. Therefore, the rest of them, i.e. $> \frac{L}{200}$ lines, contain $\geq \frac{1}{200}SkL^{-1}$ points of $\mathfrak{G} \setminus \mathfrak{G}_1$ each. So, there are $\geq \frac{L}{200} \cdot \frac{1}{200}SkL^{-1} = \frac{1}{4} \cdot 10^{-4}Sk > 2 \cdot 10^{-8}Sk$ incidences between $\mathfrak{G} \setminus \mathfrak{G}_1$ and $\mathfrak{L}$, which is not true.

Thus, $|\mathfrak{L}_1| \geq \frac{L}{200}$.

Now: since each point of $\mathfrak{G}_1$ lies in Z, it is either a critical of Z or a regular point of Z. In other words, let $\mathfrak{G}_{crit}$ be the set of critical points of Z in $\mathfrak{G}_1$ and $\mathfrak{G}_{reg}$ the set of regular points of Z in $\mathfrak{G}_1$; then, $\mathfrak{G}_1 = \mathfrak{G}_{crit} \sqcup \mathfrak{G}_{reg}$.

We know that each line of $\mathfrak{L}_1$ contains $\geq \frac{1}{200}SkL^{-1}$ points of $\mathfrak{G}_1$. Therefore, it either contains $\geq \frac{1}{400}SkL^{-1}$ points of $\mathfrak{G}_{crit}$ or $\geq \frac{1}{400}SkL^{-1}$ points of $\mathfrak{G}_{reg}$. But $|\mathfrak{L}_1| \geq \frac{L}{200}$, so, if $\mathfrak{L}_{crit}$ is the set of lines in $\mathfrak{L}_1$ each containing $\geq \frac{1}{400}SkL^{-1}$ points of $\mathfrak{G}_{crit}$ and $\mathfrak{L}_{reg}$ is the set of lines in $\mathfrak{L}_1$ each containing $\geq \frac{1}{400}SkL^{-1}$ points of $\mathfrak{G}_{reg}$, then either $|\mathfrak{L}_{crit}| \geq \frac{L}{400}$ or $|\mathfrak{L}_{reg}| \geq \frac{L}{400}$.

Now, suppose that in fact $|\mathfrak{L}_{reg}| \geq \frac{L}{400}$. This means that the incidences between $\mathfrak{L}$ and points in $\mathfrak{G}$ which are regular points of Z number at least $\frac{L}{400} \cdot \frac{1}{400}SkL^{-1} = \frac{1}{16 \cdot 10^4} \cdot Sk$. However, there exist at most $\frac{10^{-8}}{4}S$ points of $\mathfrak{G}$ which are regular points of Z, and therefore they contribute at most $\frac{10^{-8}}{4}S \cdot 2k = \frac{1}{2 \cdot 10^8}Sk \lneq \frac{1}{16 \cdot 10^4} \cdot Sk$ incidences with $\mathfrak{L}$; so, we are led to a contradiction. Therefore, $|\mathfrak{L}_{reg}| \lneq \frac{L}{400}$.

Thus, $|\mathfrak{L}_{crit}| \geq \frac{L}{400}$. Now, each line of $\mathfrak{L}_{crit}$ contains $\geq \frac{1}{400}SkL^{-1} \gneq d$ critical points of Z, i.e. $\gneq d$ points where p and ∇p are zero. However, both p and ∇p have degrees $\leq d$. Therefore, if $l \in \mathfrak{L}_{crit}$, then p and ∇p are zero across the whole line l, so each point of l is a critical point of Z; in other words, l is a critical line of Z. So, the number of critical lines of Z is $\geq |\mathfrak{L}_{crit}| \geq \frac{L}{400}$. On the other hand, the number of critical lines of Z is $\leq d^2$ (Proposition 2.2.2). So:

$$\frac{L}{400} \leq d^2,$$

which gives: $S \lesssim L^{3/2}k^{-3}$, after rearranging. Thus, $S \cdot N^{1/2} \lesssim L^{3/2}k^{-3/2}$ ($\lesssim L^{3/2}k^{-1/2}$).

In other words,

$$SN^{1/2} \leq c_3 \cdot \frac{L^{3/2}}{k^{1/2}},$$

for some constant c_3 independent of $\mathfrak{L}$, N and k.

• **If $|\mathfrak{L}'| < \frac{L}{100}$:**

Since at most $\frac{10^{-8}}{4}$ points of $\mathfrak{G}_1$ are regular points of Z, the same holds for the subset $\mathfrak{G}'$ of $\mathfrak{G}_1$. So, at least $\frac{1-2\cdot 10^{-8}-10^{-2}}{4}S$ points of $\mathfrak{G}'$ are critical points of Z.

Now, each of the points of $\mathfrak{G}'$ is a joint for $\mathfrak{L}$ with multiplicity in the interval $[N, 2N)$, so it is either a joint for $\mathfrak{L}'$ with multiplicity in the interval $[N/2, 2N)$, or it is a joint for $\mathfrak{L}'$ with multiplicity $< N/2$, or it is not a joint for $\mathfrak{L}'$. Therefore, one of the following two subcases holds:

1st subcase: There exist at least $\frac{1-2\cdot 10^{-8}-10^{-2}}{8}S$ critical points in $\mathfrak{G}'$ each of which is either a joint for $\mathfrak{L}'$ with multiplicity $< N/2$ or not a joint at all for $\mathfrak{L}'$. Let $\mathfrak{G}_2$ be the set of those points.

By Lemma 3.1, for each point $x \in \mathfrak{G}_2$ there exist $\geq \frac{N}{1000 \cdot k^2}$ lines of $\mathfrak{L} \setminus \mathfrak{L}'$ passing through x.

Now, let $\mathfrak{L}_3$ be the set of lines in $\mathfrak{L} \setminus \mathfrak{L}'$, such that each of them contains $\leq d$ critical points. Then, one of the following two holds:

(1) There exist $\geq \frac{1-2\cdot 10^{-8}-10^{-2}}{16} S$ points of $\mathfrak{G}_2$ such that each of them has $\geq \frac{N}{2000 \cdot k^2}$ lines of $\mathfrak{L}_3$ passing through it. Then,

$$S \cdot \frac{N}{k^2} \lesssim I_{\mathfrak{G}_2, \mathfrak{L}_3} \leq |\mathfrak{L}_3| \cdot d \leq L \cdot d.$$

Rearranging, we get that $S \cdot N^{1/2} \lesssim L^{3/2} k^{-1/2}$.

(2) There exist $\geq \frac{1-2\cdot 10^{-8}-10^{-2}}{16} S$ points of $\mathfrak{G}_2$ such that each of them has $\geq \frac{N}{2000 \cdot k^2}$ lines of $\big(\mathfrak{L} \setminus \mathfrak{L}'\big) \setminus \mathfrak{L}_3$ passing through it. Each line of $\big(\mathfrak{L} \setminus \mathfrak{L}'\big) \setminus \mathfrak{L}_3$ contains $< \frac{1}{100} SkL^{-1}$ points of $\mathfrak{G}_1$. Also, it contains $> d$ critical points of Z, so it is a critical line. But Z contains $\leq d^2$ critical lines in total (by Proposition 2.2.2). Therefore,

$$S \cdot \frac{N}{k^2} \lesssim I_{\mathfrak{G}_2, \big(\mathfrak{L} \setminus \mathfrak{L}'\big) \setminus \mathfrak{L}_3} \leq d^2 \cdot \frac{1}{100} SkL^{-1},$$

so $S \cdot N^{1/2} \lesssim L^{3/2} k^{-1/2}$, by rearranging.

Thus, in this 1st subcase,

$$S \cdot N^{1/2} \leq c_4 \cdot \frac{L^{3/2}}{k^{1/2}},$$

where c_4 is a constant independent of $\mathfrak{L}$, N and k.

We are now able to define the constant c appearing in our induction process: we let $c := \max\{2c_0, c_1, c_2, c_3, c_4\}$. This way, in any case that has been dealt with so far,

$$S \cdot N^{1/2} \leq c \cdot \left(\frac{L^{3/2}}{k^{1/2}} + \frac{L}{k} \cdot N^{1/2} \right).$$

Note that c is, indeed, an explicit, non-negative constant, independent of $\mathfrak{L}$, N and k.

2nd subcase: At least $\frac{1-2\cdot 10^{-8}-10^{-2}}{8} S$ points of $\mathfrak{G}'$ are joints for $\mathfrak{L}'$ with multiplicity in the interval $[\frac{N}{2}, 2N)$. Then, either **(1)** or **(2)** hold:

(1) At least $\frac{1-2\cdot 10^{-8}-10^{-2}}{16} S$ points of $\mathfrak{G}'$ are joints for $\mathfrak{L}'$ with multiplicity in the interval $[N, 2N)$. However, each point of $\mathfrak{G}'$ intersects between $\frac{1-10^{-8}-10^{-2}}{2} k$ and $2k$ lines of $\mathfrak{L}'$. Therefore, either $(1i)$, $(1ii)$ or $(1iii)$ hold.

$(1i)$ At least $\frac{1-2\cdot 10^{-8}-10^{-2}}{48} S$ points of $\mathfrak{G}'$ are joints for $\mathfrak{L}'$ with multiplicity in the interval $[N, 2N)$, such that each of them lies in between k and $2k$ lines of $\mathfrak{L}'$. Then, since $|\mathfrak{L}'| < \frac{L}{100} \lneq L$, we get from our induction hypothesis that

$$\frac{1-2\cdot 10^{-8}-10^{-2}}{48} S\cdot N^{1/2} \leq c\cdot\left(\frac{|\mathfrak{L}'|^{3/2}}{k^{1/2}}+\frac{|\mathfrak{L}'|}{k}\cdot N^{1/2}\right)\leq$$

$$\leq c\cdot\left(\frac{(L/100)^{3/2}}{k^{1/2}}+\frac{(L/100)}{k}\cdot N^{1/2}\right).$$

But

$$\frac{48}{1-2\cdot 10^{-8}-10^{-2}}\cdot\frac{1}{100^{3/2}}<1$$

and

$$\frac{48}{1-2\cdot 10^{-8}-10^{-2}}\cdot\frac{1}{100}<1,$$

therefore

$$S\cdot N^{1/2}\leq c\cdot\left(\frac{L^{3/2}}{k^{1/2}}+\frac{L}{k}\cdot N^{1/2}\right).$$

(1ii) At least $\frac{1-2\cdot 10^{-8}-10^{-2}}{48}S$ points of $\mathfrak{G}'$ are joints for $\mathfrak{L}'$ with multiplicity in the interval $[N, 2N)$, such that each of them lies in between $\frac{1-10^{-8}-10^{-2}}{2}k$ and $(1-10^{-8}-10^{-2})k$ lines of $\mathfrak{L}'$. So, since $|\mathfrak{L}'|<\frac{L}{100}\lneq L$, we get from our induction hypothesis that

$$\frac{1-2\cdot 10^{-8}-10^{-2}}{48} S\cdot N^{1/2}\leq c\cdot\left(\frac{|\mathfrak{L}'|^{3/2}}{\left(\frac{1-10^{-8}-10^{-2}}{2}k\right)^{1/2}}+\frac{|\mathfrak{L}'|}{\left(\frac{1-10^{-8}-10^{-2}}{2}k\right)}\cdot N)^{1/2}\right)\leq$$

$$\leq c\cdot\left(\frac{(L/100)^{3/2}}{\left(\frac{1-10^{-8}-10^{-2}}{2}k\right)^{1/2}}+\frac{(L/100)}{\left(\frac{1-10^{-8}-10^{-2}}{2}k\right)}\cdot N^{1/2}\right).$$

But

$$\frac{48}{1-2\cdot 10^{-8}-10^{-2}}\cdot\frac{1}{100^{3/2}}\cdot\frac{2^{1/2}}{(1-10^{-8}-10^{-2})^{1/2}}<1$$

and

$$\frac{48}{1-2\cdot 10^{-8}-10^{-2}}\cdot\frac{1}{100}\cdot\frac{2}{1-10^{-8}-10^{-2}}<1,$$

therefore

$$S\cdot N^{1/2}\leq c\cdot\left(\frac{L^{3/2}}{k^{1/2}}+\frac{L}{k}\cdot N^{1/2}\right).$$

(1iii) At least $\frac{1-2\cdot 10^{-8}-10^{-2}}{48}S$ points of $\mathfrak{G}'$ are joints for $\mathfrak{L}'$ with multiplicity in the interval $[N, 2N)$, such that each of them lies in between $(1-10^{-8}-10^{-2})k$ and $2\cdot(1-10^{-8}-10^{-2})k$ lines of $\mathfrak{L}'$. So, since $|\mathfrak{L}'|<\frac{L}{100}\lneq L$, we get from our induction hypothesis that

$$\frac{1-2\cdot 10^{-8}-10^{-2}}{48}S\cdot N^{1/2}\le c\cdot\left(\frac{|\mathfrak{L}'|^{3/2}}{((1-10^{-8}-10^{-2})k)^{1/2}}+\frac{|\mathfrak{L}'|}{(1-10^{-8}-10^{-2})k}\cdot N^{1/2}\right)\le$$

$$\le c\cdot\left(\frac{(L/100)^{3/2}}{\big((1-10^{-8}-10^{-2})k\big)^{1/2}}+\frac{(L/100)}{(1-10^{-8}-10^{-2})k}\cdot N^{1/2}\right).$$

But

$$\frac{48}{1-2\cdot 10^{-8}-10^{-2}}\cdot\frac{1}{100^{3/2}}\cdot\frac{1}{(1-10^{-8}-10^{-2})^{1/2}}<1$$

and

$$\frac{48}{1-2\cdot 10^{-8}-10^{-2}}\cdot\frac{1}{100}\cdot\frac{1}{1-10^{-8}-10^{-2}}<1,$$

therefore

$$S\cdot N^{1/2}\le c\cdot\left(\frac{L^{3/2}}{k^{1/2}}+\frac{L}{k}\cdot N^{1/2}\right).$$

(2) At least $\frac{1-2\cdot 10^{-8}-10^{-2}}{16}S$ points of $\mathfrak{G}'$ are joints for $\mathfrak{L}'$ with multiplicity in the interval $[\frac{N}{2},N)$. However, each point of $\mathfrak{G}'$ intersects between $\frac{1-10^{-8}-10^{-2}}{2}\cdot k$ and $2k$ lines of $\mathfrak{L}'$. Therefore, either $(2i)$, $(2ii)$ or $(2iii)$ hold.

$(2i)$ At least $\frac{1-2\cdot 10^{-8}-10^{-2}}{48}S$ points of $\mathfrak{G}'$ are joints for $\mathfrak{L}'$ with multiplicity in the interval $[\frac{N}{2},N)$, such that each of them lies in between k and $2k$ lines of $\mathfrak{L}'$. Then, since $|\mathfrak{L}'|<\frac{L}{100}\lneq L$, we get from our induction hypothesis that

$$\frac{1-2\cdot 10^{-8}-10^{-2}}{48}S\cdot\left(\frac{N}{2}\right)^{1/2}\le c\cdot\left(\frac{|\mathfrak{L}'|^{3/2}}{k^{1/2}}+\frac{|\mathfrak{L}'|}{k}\cdot\left(\frac{N}{2}\right)^{1/2}\right)\le$$

$$\le c\cdot\left(\frac{(L/100)^{3/2}}{k^{1/2}}+\frac{(L/100)}{k}\cdot\left(\frac{N}{2}\right)^{1/2}\right).$$

But

$$\frac{48}{1-2\cdot 10^{-8}-10^{-2}}\cdot\frac{1}{100^{3/2}}\cdot 2^{1/2}<1$$

and

$$\frac{48}{1-2\cdot 10^{-8}-10^{-2}}\cdot\frac{1}{100}<1,$$

therefore

$$S\cdot N^{1/2}\le c\cdot\left(\frac{L^{3/2}}{k^{1/2}}+\frac{L}{k}\cdot N^{1/2}\right).$$

$(2ii)$ At least $\frac{1-2\cdot 10^{-8}-10^{-2}}{48}S$ points of $\mathfrak{G}'$ are joints for $\mathfrak{L}'$ with multiplicity in the interval $[\frac{N}{2}, N)$, such that each of them lies in between $\frac{1-10^{-8}-10^{-2}}{2}\cdot k$ and $(1-10^{-8}-10^{-2})k$ lines of $\mathfrak{L}'$. So, since $|\mathfrak{L}'| < \frac{L}{100} \lneq L$, we get from our induction hypothesis that

$$\frac{1-2\cdot 10^{-8}-10^{-2}}{48}S\cdot\left(\frac{N}{2}\right)^{1/2} \leq c\cdot\left(\frac{|\mathfrak{L}'|^{3/2}}{\left(\frac{1-10^{-8}-10^{-2}}{2}k\right)^{1/2}}+\frac{|\mathfrak{L}'|}{\left(\frac{1-10^{-8}-10^{-2}}{2}k\right)}\cdot\left(\frac{N}{2}\right)^{1/2}\right) \leq$$

$$\leq c\cdot\left(\frac{(L/100)^{3/2}}{\left(\frac{1-10^{-8}-10^{-2}}{2}k\right)^{1/2}}+\frac{(L/100)}{\left(\frac{1-10^{-8}-10^{-2}}{2}k\right)}\cdot\left(\frac{N}{2}\right)^{1/2}\right).$$

But

$$\frac{48}{1-2\cdot 10^{-8}-10^{-2}}\cdot\frac{1}{100^{3/2}}\cdot\frac{2^{1/2}}{(1-10^{-8}-10^{-2})^{1/2}}\cdot 2^{1/2} < 1$$

and

$$\frac{48}{1-2\cdot 10^{-8}-10^{-2}}\cdot\frac{1}{100}\cdot\frac{2}{1-10^{-8}-10^{-2}} < 1,$$

therefore

$$S\cdot N^{1/2} \leq c\cdot\left(\frac{L^{3/2}}{k^{1/2}}+\frac{L}{k}\cdot N^{1/2}\right).$$

$(2iii)$ At least $\frac{1-2\cdot 10^{-8}-10^{-2}}{48}S$ points of $\mathfrak{G}'$ are joints for $\mathfrak{L}'$ with multiplicity in the interval $[\frac{N}{2}, N)$, such that each of them lies in between $(1-10^{-8}-10^{-2})k$ and $2\cdot(1-10^{-8}-10^{-2})k$ lines of $\mathfrak{L}'$. So, since $|\mathfrak{L}'| < \frac{L}{100} \lneq L$, we get from our induction hypothesis that

$$\frac{1-2\cdot 10^{-8}-10^{-2}}{48}S\cdot\left(\frac{N}{2}\right)^{1/2} \leq c\cdot\left(\frac{|\mathfrak{L}'|^{3/2}}{((1-10^{-8}-10^{-2})k)^{1/2}}+\frac{|\mathfrak{L}'|}{(1-10^{-8}-10^{-2})k}\cdot\left(\frac{N}{2}\right)^{1/2}\right) \leq$$

$$\leq c\cdot\left(\frac{(L/100)^{3/2}}{\left((1-10^{-8}-10^{-2})k\right)^{1/2}}+\frac{(L/100)}{(1-10^{-8}-10^{-2})k}\cdot\left(\frac{N}{2}\right)^{1/2}\right).$$

But

$$\frac{48}{1-2\cdot 10^{-8}-10^{-2}}\cdot\frac{1}{100^{3/2}}\cdot\frac{2^{1/2}}{(1-10^{-8}-10^{-2})^{1/2}} < 1$$

and

$$\frac{48}{1-2\cdot 10^{-8}-10^{-2}}\cdot\frac{1}{100}\cdot\frac{1}{1-10^{-8}-10^{-2}} < 1,$$

therefore

$$S \cdot N^{1/2} \leq c \cdot \left(\frac{L^{3/2}}{k^{1/2}} + \frac{L}{k} \cdot N^{1/2} \right).$$

We have by now exhausted all the possible cases; in each one,

$$S \cdot N^{1/2} \leq c \cdot \left(\frac{L^{3/2}}{k^{1/2}} + \frac{L}{k} \cdot N^{1/2} \right),$$

where c is, by its definition, a constant independent of $\mathfrak{L}$, N and k.

Therefore, (4) holds for this selection of $\mathfrak{L}$, N and k. As N and k were arbitrary, (4) holds $\forall N, k \in \mathbb{N}$, for this collection $\mathfrak{L}$ of lines. And since $\mathfrak{L}$ was an arbitrary collection of L lines, (4) holds for any collection $\mathfrak{L}$ of L lines, $\forall N, k \in \mathbb{N}$.

Consequently, the proposition is proved.

□

Now, Theorem 1.1 will easily follow:

Theorem 1.1: *Let $\mathfrak{L}$ be a collection of L lines in $\mathbb{R}^3$, forming a set of joints J. Then,*

$$\sum_{x \in J} N(x)^{1/2} \leq c \cdot L^{3/2},$$

where c is a constant independent of $\mathfrak{L}$.

Proof. The multiplicity of each joint in J can be at most $\binom{L}{3} \leq L^3$. Therefore,

$$\sum_{x \in J} N(x)^{1/2} \leq 2 \cdot \sum_{\lambda \in \mathbb{N}: 2^\lambda \leq L^3} |J_{2^\lambda}| \cdot (2^\lambda)^{1/2}.$$

But if x is a joint for $\mathfrak{L}$ with multiplicity N, such that $\leq 2k$ lines of $\mathfrak{L}$ are passing through x, then $N \leq \binom{2k}{3} \leq (2k)^3$. Thus, $\forall \lambda \in \mathbb{N}$ such that $2^\lambda \leq L^3$,

$$|J_{2^\lambda}| = \sum_{\mu \in \mathbb{N}: 2^\mu \geq \frac{1}{2}(2^\lambda)^{1/3}} |J_{2^\lambda}^{2^\mu}|,$$

so

$$|J_{2^\lambda}| \cdot (2^\lambda)^{1/2} = \sum_{\mu \in \mathbb{N}: 2^\mu \geq \frac{1}{2}(2^\lambda)^{1/3}} |J_{2^\lambda}^{2^\mu}| \cdot (2^\lambda)^{1/2},$$

a quantity which, from Proposition 1.2, is

$$\leq \sum_{\mu \in \mathbb{N}: 2^\mu \geq \frac{1}{2}(2^\lambda)^{1/3}} c \cdot \left(\frac{L^{3/2}}{(2^\mu)^{1/2}} + \frac{L}{2^\mu} \cdot (2^\lambda)^{1/2} \right) \leq$$

$$\leq c' \cdot \left(\frac{L^{3/2}}{\left((2^\lambda)^{1/3}\right)^{1/2}} + \frac{L}{(2^\lambda)^{1/3}} \cdot (2^\lambda)^{1/2} \right),$$

where c' is a constant independent of $\mathfrak{L}$, k and λ.

Therefore,

$$\sum_{x \in J} N(x)^{1/2} \leq 2c' \cdot \sum_{\lambda \in \mathbb{N}: 2^\lambda \leq L^3} \left(\frac{L^{3/2}}{(2^\lambda)^{1/6}} + L \cdot (2^\lambda)^{1/6} \right) \leq c'' \cdot \left(L^{3/2} + L \cdot L^{1/2}\right) = c'' \cdot L^{3/2},$$

where c'' is a constant independent of $\mathfrak{L}$.

The proof of Theorem 1.1 is now complete.

□

4. The case of more general curves

In this section we extend the definition of a joint to a more general setting. More particularly, let $\mathcal{F}$ be the family of all non-empty sets in $\mathbb{R}^3$ with the property that, if $\gamma \in \mathcal{F}$ and $x \in \gamma$, then a basic neighbourhood of x in γ is either $\{x\}$ or the finite union of parametric curves, each analytically homeomorphic to a semi-open line segment with one endpoint the point x. In addition, if there exists a parametrisation $f : [0,1) \to \mathbb{R}^3$ of one of these curves, with $f(0) = x$ and $f'(0) \neq 0$, then the line in $\mathbb{R}^3$ passing through x with direction $f'(0)$ is tangent to γ at x. If $\Gamma \subset \mathcal{F}$, we denote by T_x^Γ the set of directions of all tangent lines at x to the sets of Γ passing through x (note that T_x^Γ might be empty and that there might exist many tangent lines to a set of Γ at x).

Definition: *Let Γ be a collection of sets in $\mathcal{F}$. Then a point x in $\mathbb{R}^3$ is a joint for the collection Γ if:*

(i) x belongs to at least one of the sets in Γ, and

(ii) there exist at least 3 vectors in T_x^Γ spanning $\mathbb{R}^3$.

The multiplicity $N(x)$ of the joint x is defined as the number of triples of linearly independent vectors in T_x^Γ.

We will show here that, under certain assumptions on the characteristics of the sets in a finite collection $\Gamma \subset \mathcal{F}$, the statement of Theorem 1.1 still holds, i.e.

$$\sum_{x \in J} N(x)^{1/2} \leq c \cdot |\Gamma|^{3/2},$$

where J is the set of joints formed by Γ. To that end, we will need to recall and further analyse some facts from algebraic geometry.

4.1. **Analysing the geometric background.**

If $\mathbb{K}$ is a field, any set of the form

$$\{x \in \mathbb{K}^n : p_i(x) = 0, \ \forall\, i = 1, ..., k\},$$

where $p_i \in \mathbb{K}[x_1, ..., x_n]\ \forall\ i = 1, ..., k$, is called an *algebraic set* or an *affine variety* or simply a *variety* in $\mathbb{K}^n$, and is denoted by $V(p_1, ..., p_k)$. A variety V in $\mathbb{K}^n$ is *irreducible* if it cannot be expressed as the union of two non-empty varieties in $\mathbb{K}^n$ which are strict subsets of V.

Now, if V is a variety in $\mathbb{K}^n$, the set

$$I(V) := \{p \in \mathbb{K}[x_1, ..., x_n] : p(x) = 0,\ \forall\ x \in V\}$$

is an ideal in $\mathbb{K}[x_1, ..., x_n]$. If, in particular, V is irreducible, then $I(V)$ is a prime ideal of $\mathbb{K}[x_1, ..., x_n]$, and the transcendence degree of the ring $\mathbb{K}[x_1, ..., x_n]/I(V)$ over $\mathbb{K}$ is the *dimension* of the irreducible variety V. The *dimension of an algebraic set* is the maximal dimension of all the irreducible varieties contained in the set. If an algebraic set has dimension 1 it is called an *algebraic curve*, while if it has dimension $n-1$ it is called an *algebraic surface*. Note that all irreducible algebraic curves in $\mathbb{R}^3$ are contained in the family $\mathcal{F}$ defined above.

Now, an order $\prec$ on the set of monomials in $\mathbb{K}[x_1, ..., x_n]$ is called a *term order*, if it is a total order on the monomials of $\mathbb{K}[x_1, ..., x_n]$, such that it is multiplicative (i.e. it is preserved by multiplication by the same monomial) and the constant monomial is the $\prec$-smallest monomiall. Then, if I is an ideal in $\mathbb{K}[x_1, ..., x_n]$, we define $in_\prec(I)$ as the ideal of $\mathbb{K}[x_1, ..., x_n]$ consisting of the $\prec$-initial terms, i.e. the $\prec$-largest monomial terms, of all the polynomials in I.

Let V be a variety in $\mathbb{K}[x_1, ..., x_n]$ and $\prec$ a term order on the set of monomials in $\mathbb{K}[x_1, ..., x_n]$. Also, let S be a maximal subset of the set of variables $\{x_1, ..., x_n\}$, with the property that no monomial in the variables in S belongs to $in_\prec(I(V))$. Then, it holds that the dimension of V is the cardinality of S (see [Stu05]). From this fact, we deduce the following:

Lemma 4.1.1. *An irreducible real algebraic curve γ of $\mathbb{R}^n$ is contained in an irreducible complex algebraic curve of $\mathbb{C}^n$.*

Proof. We clarify that, by saying that a real algebraic curve γ_1 in $\mathbb{R}^n$ is contained in a complex algebraic curve γ_2 in $\mathbb{C}^n$, we mean that, if $x \in \gamma_1$, then the point x, seen as an element of $\mathbb{C}^n$, belongs to γ_2 as well.

Let γ be an irreducible real algebraic curve in $\mathbb{R}^n$, and $\prec$ a term order on the set of monomials in variables $\{x_1, ..., x_n\}$. From the discussion above, for every $i \neq j$, $i, j \in \{1, ..., n\}$, there exists a monomial in the variables x_i and x_j in the ideal $in_\prec(I(\gamma))$. The ideal $I(\gamma)$ is finitely generated, like any ideal of $\mathbb{R}[x_1, ..., x_n]$. Let $\{p_1, ..., p_k\}$ be a finite set of generators of $I(\gamma)$, and let $I' := (p_1, ..., p_k)$ be the ideal in $\mathbb{C}[x_1, ..., x_n]$ generated by the polynomials p_1, ..., p_k, this time seen as elements of $\mathbb{C}[x_1, ..., x_n]$. We consider the complex variety $V' = V(p_1, ..., p_k)$ and the ideal $I(V')$ in $\mathbb{C}[x_1, ..., x_n]$. Since the polynomials in $I(\gamma)$, seen as elements of $\mathbb{C}[x_1, ..., x_n]$, are elements of $I(V')$, it holds that for every $i \neq j$, $i, j \in \{1, ..., n\}$, there exists a monomial in the variables x_i and x_j in the ideal $in_\prec(I(V'))$. Therefore, the variety V' has dimension 1 (it cannot have dimension 0, as it is not a finite set of points). Therefore, we have proved that γ is contained in a complex algebraic curve.

It is finally easy to see that it is contained in an irreducible component of that curve.

$\square$

Now, if γ is an algebraic curve in $\mathbb{C}^n$, a generic hyperplane of $\mathbb{C}^n$ intersects the curve in a specific number of points (counted with appropriate multiplicities), which is called the *degree* of the curve. On the other hand, this is not true in general for real algebraic curves. However, by Lemma 4.1.1, we can define the *degree of a real algebraic curve* in $\mathbb{R}^n$ as the degree of the smallest complex algebraic curve in $\mathbb{C}^n$ containing it. Therefore, if, by saying that a real algebraic curve γ in $\mathbb{R}^n$ *crosses itself at the point* $x_0 \in \gamma$, we mean that any neighbourhood of x_0 in γ is neither $\{x_0\}$ nor analytically homeomorphic to $\mathbb{R}$, it follows that a real algebraic curve in $\mathbb{R}^n$ crosses itself at a point at most as many times as its degree.

We now state a consequence of Bézout's theorem (see, for example, [Ful84, Theorem 12.3] or [CLO05, Chapter 3, §3]):

Theorem 4.1.2. (Bézout) *Let γ be an irreducible algebraic curve in $\mathbb{C}^n$ of degree b, and $p \in \mathbb{C}[x_1, ..., x_n]$. If γ is not contained in the zero set of p, it intersects the zero set of p at most $b \cdot \deg p$ times.*

An immediate consequence of this theorem and the discussion above is the following:

Corollary 4.1.3. *Let γ be an irreducible algebraic curve in $\mathbb{R}^n$ of degree b, and $p \in \mathbb{R}[x_1, ..., x_n]$. If γ is not contained in the zero set of p, it intersects the zero set of p at most $b \cdot \deg p$ times.*

Theorem 4.1.2 also implies:

Corollary 4.1.4. *Let γ_1, γ_2 be two distinct irreducible complex algebraic curves in $\mathbb{C}^n$. Then, they have $\lesssim_{\deg \gamma_1, \deg \gamma_2, n} 1$ common points.*

Proof. Since γ_2 is an algebraic curve in $\mathbb{C}^n$ and $\mathbb{C}$ is an algebraically closed field, it follows that γ_2 is the intersection of the zero sets of $\lesssim_{n, \deg \gamma_2} 1$ irreducible polynomials in $\mathbb{C}[x_1, ..., x_n]$, each of which has degree at most $\deg \gamma_2$ (see [BGT11, Theorem A.3]). The zero set of at least one of these polynomials does not contain γ_1, so by Theorem 4.1.2 γ_1 intersects it $\lesssim_{\deg \gamma_1, \deg \gamma_2, n} 1$ times. Therefore, γ_1 intersects γ_2, which is contained in the zero set of the above-mentioned polynomial, $\lesssim_{\deg \gamma_1, \deg \gamma_2, n} 1$ times.

$\square$

We now discuss projections of algebraic curves. More particularly, a *basic real semi-algebraic set* in $\mathbb{R}^n$ is any set of the form

$$\{x \in \mathbb{R}^n : P(x) = 0 \text{ and } Q(x) > 0, \ \forall \ Q \in \mathcal{Q}\},$$

where $P \in \mathbb{R}[x_1, ..., x_n]$ and $\mathcal{Q}$ is a finite family of polynomials in $\mathbb{R}[x_1, ..., x_n]$. A *real semi-algebraic set* in $\mathbb{R}^n$ is defined as a finite union of basic real semi-algebraic sets. Note that a real algebraic set in $\mathbb{R}^n$ is, in fact, a basic real semi-algebraic set in $\mathbb{R}^n$, since it can be expressed as the zero set of a single

real polynomial. What holds is the following (see [BPR06, Chapter 2, §3] for a proof):

Theorem 4.1.5. *The projection of a real algebraic set of* $\mathbb{R}^n$ *on any hyperplane of* $\mathbb{R}^n$ *is a real semi-algebraic set.*

We further notice that any set of the form $\{x \in \mathbb{R}^n : Q(x) > 0,\ \forall\, Q \in \mathcal{Q}\}$, where $\mathcal{Q}$ is a finite subset of $\mathbb{R}[x_1, ..., x_n]$, is open in $\mathbb{R}^n$ (with the usual topology). Therefore, a basic real semi-algebraic set of $\mathbb{R}^n$ that is not open in $\mathbb{R}^n$ is of the form $\{x \in \mathbb{R}^n : P(x) = 0 \text{ and } Q(x) > 0,\ \forall\, Q \in \mathcal{Q}\}$, where $\mathcal{Q}$ is a finite subset of $\mathbb{R}[x_1, ..., x_n]$ and $P \in \mathbb{R}[x_1, ..., x_n]$ is a non-zero polynomial. Thus, each basic real semi-algebraic set of $\mathbb{R}^n$ that is not open in $\mathbb{R}^n$ (with the usual topology) is contained in a real algebraic set of dimension at most $n-1$.

Now, if γ is a real algebraic curve in $\mathbb{R}^3$, its projection on a generic plane $H \simeq \mathbb{R}^2$ is a finite union of basic real semi-algebraic sets which are not open in H, so each of them is contained in some real algebraic set of dimension at most 1. However, the projection of a curve of $\mathbb{R}^3$ on a generic plane is not a finite set of points. Therefore, at least one of these basic real semi-algebraic sets is contained in some real algebraic curve in H. From this fact, as well as a closer study of the algorithm that constitutes the proof of Theorem 4.1.5 as described in [BPR06, Chapter 2, §3], we can finally see that the projection of γ on a generic plane is the union of at most $B_{\deg\gamma}$ basic real semi-algebraic sets, each of which either consists of at most $B'_{\deg\gamma}$ points or is contained in a real algebraic curve in H of degree at most $B'_{\deg\gamma}$, where $B_{\deg\gamma}$, $B'_{\deg\gamma}$ are integers depending only on the degree $\deg\gamma$ of γ. Therefore, the following is true:

Lemma 4.1.6. *Let* γ *be a real algebraic curve in* $\mathbb{R}^3$*. Its projection on a generic plane is contained in a real algebraic curve of degree at most* $C_{\deg\gamma}$*, where* $C_{\deg\gamma} \geq \deg\gamma$ *is an integer depending only on the degree* $\deg\gamma$ *of* γ*.*

Note that this means that the *Zariski closure* of the projection of γ on a generic plane, i.e. the smallest variety containing that projection, is, in fact, an algebraic curve.

In addition, the following holds:

Lemma 4.1.7. *Let* γ *be a real algebraic curve in* $\mathbb{R}^3$*. Let* $\pi(\gamma)$ *be the projection of the curve on a generic plane and* $\overline{\pi(\gamma)}$ *the smallest planar real algebraic curve containing this projection (i.e. the curve that constitutes the Zariski closure of* $\pi(\gamma)$*). Then,* γ *crosses itself at most* $(\deg\overline{\pi(\gamma)})^2$ *times.*

Proof. Obviously, γ crosses itself at most as many times as $\overline{\pi(\gamma)}$ crosses itself. Now, $\overline{\pi(\gamma)}$ is the zero set of a square-free polynomial $p \in \mathbb{R}[x_1, x_2]$ of degree at most $\deg\overline{\pi(\gamma)}$. Then, p and ∇p do not have a common factor, so, by Bézout's theorem, p and ∇p have at most $(\deg p)^2 \leq (\deg\overline{\pi(\gamma)})^2$ common roots. But if $\overline{\pi(\gamma)}$ crosses itself at a point x, then x is a common root of p and ∇p, because otherwise $\overline{\pi(\gamma)}$ would be locally a manifold around x. So,

$\overline{\pi(\gamma)}$ crosses itself at most $(\deg \overline{\pi(\gamma)})^2$ times, therefore so does γ.

□

We are now ready to establish an analogue of the Szemerédi - Trotter theorem for real algebraic curves in $\mathbb{R}^3$. Indeed, it is known that:

Theorem 4.1.8. (Kaplan, Matoušek, Sharir, [KMS11, Theorem 4.1]) *Let b, k, C be positive constants. Also, let P be a finite set of points in $\mathbb{R}^2$ and Γ a finite set of real planar algebraic curves, such that*

(i) every $\gamma \in \Gamma$ has degree at most b, and

(ii) for every k distinct points in $\mathbb{R}^2$, there exist at most C distinct curves in Γ passing through all of them.

Then,

$$I_{P,\Gamma} \lesssim_{b,k,C} |P|^{k/(2k-1)}|\Gamma|^{(2k-2)/(2k-1)} + |P| + |\Gamma|.$$

Combining Theorem 4.1.8 with Lemmas 4.1.6 and 4.1.7, we deduce the following:

Lemma 4.1.9. *Let b be a positive constant. Also, let Γ be a finite set of real algebraic curves in $\mathbb{R}^3$, each of degree at most b, and let P be a finite set of points in $\mathbb{R}^3$. Then, there exists a natural number $D_b \geq b$, depending only on b, such that*

(i) $I'_{P,\Gamma} \lesssim_b |P|^{D_b/(2D_b-1)}|\Gamma|^{(2D_b-2)/(2D_b-1)} + |P| + |\Gamma|$, where $I'_{P,\Gamma}$ denotes the number of all pairs (p, γ) such that $p \in P$, $\gamma \in \Gamma$, $p \in \gamma$ and p is not an isolated point of γ, and

(ii) if there exist S points in $\mathbb{R}^3$, such that each lies in at least k curves of Γ which do not have the point as an isolated point, then $S \lesssim_b |\Gamma|^2/k^{(2D_b-1)/(D_b-1)} + |\Gamma|/k$.

Proof. Let $\pi : \mathbb{R}^3 \to H$ the projection map of $\mathbb{R}^3$ on a generic plane $H \simeq \mathbb{R}^2$. By Lemma 4.1.6 we know that, for all $\gamma \in \Gamma$, $\pi(\gamma)$ is contained in a real planar algebraic curve $\overline{\pi(\gamma)}$ of degree at most C_b, where $C_b \geq b$ is an integer depending only on b. Thus, if $\pi(\Gamma) = \{\pi(\gamma) : \gamma \in \Gamma\}$ and $Ext(\pi(\Gamma)) = \{$irreducible 1-dimensional components of $\overline{\pi(\gamma)} : \gamma \in \Gamma\}$, we have that

$$I'_{P,\Gamma} \leq I'_{\pi(P),\pi(\Gamma)} \leq I_{\pi(P),Ext(\pi(\Gamma))} + {C_b}^2 \cdot |Ext(\pi(\Gamma))|,$$

as each curve in $Ext(\pi(\Gamma)$ crosses itself at most $(\deg \overline{\pi(\gamma)})^2$ times. In addition, by Bézout's theorem, for each $C_b^2 + 1$ distinct points of $\mathbb{R}^2$ there exists at most 1 curve of $Ext(\pi(\Gamma))$ passing through all of them. The application, therefore, of Theorem 4.1.8 for $k = D_b := C_b^2 + 1$, the set of points $\pi(P)$ and the set of real planar algebraic curves $Ext(\pi(\Gamma))$, whose cardinality is obviously $\leq C_b \cdot |\Gamma|$, completes the proof of (i), while (ii) is an immediate corollary of (i).

□

We are also interested in curves in $\mathbb{R}^3$ parametrised by $t \to \big(p_1(t), p_2(t), p_3(t)\big)$ for $t \in \mathbb{R}$, where $p_i \in \mathbb{R}[x, y, z]$ for $i = 1, 2, 3$. Note that (see [CLO91, Chapter 3, §3]), although curves in $\mathbb{C}^3$ with a polynomial parametrisation are, in fact, complex algebraic curves of degree equal to the maximal degree of the polynomials realising the parametrisation, curves in $\mathbb{R}^3$ with a polynomial parametrisation are not, in general, real algebraic curves, which is the reason why we treat their case separately. More particularly, if a curve γ in $\mathbb{R}^3$ is parametrised by $t \to \big(p_1(t), p_2(t), p_3(t)\big)$ for $t \in \mathbb{R}$, where the $p_i \in \mathbb{R}[x, y, z]$, for $i = 1, 2, 3$, are polynomials not simultaneously constant, then the complex algebraic curve parametrised by the same polynomials seen in $\mathbb{C}[x, y, z]$ is irreducible (it is easy to see that if it contains a complex algebraic curve, then the two curves are identical). It is thus the smallest complex algebraic curve containing γ.

We now establish an analogue of Lemma 4.1.9 for real curves with a polynomial parametrisation:

Lemma 4.1.10. *Let b be a positive constant. Also, let P be a set of m points in $\mathbb{R}^2$ and Γ a family of n curves in $\mathbb{R}^2$, such that each $\gamma \in \Gamma$ is parametrised by $t \to \big(p_1^\gamma(t), p_2^\gamma(t)\big)$ for $t \in \mathbb{R}$, where the $p_i^\gamma \in \mathbb{R}[x, y]$, for $i = 1, 2$, are polynomials not simultaneously constant, of degrees at most b. Then,*

$$I_{P,\Gamma} \lesssim_b m^{b/(2b-1)} n^{(2b-2)/(2b-1)} + m + n.$$

Proof. The proof is a combination of the proofs of [KMS11, Theorem 1.1] and [KMS11, Theorem 4.1]. Indeed, there are at most $\binom{m-1}{b}$ curves of Γ with at least $b+1$ incidences passing through any fixed point of P, as, for every $b+1$ distinct points in $\mathbb{R}^2$, there exists at most 1 curve of Γ passing through all of them. So, $I_{P,\Gamma} \lesssim_b n + m^b$. Also, $I_{P,\Gamma} \lesssim_b m + n^2$ (proof as in [KMS11, Theorem 4.1]).

Therefore, we may assume that $m \leq n^2$ and $n \leq m^b$. We set $r := m^{2b/(2b-1)}/n^{2/(2b-1)}$ (≥ 1), and applying the Guth - Katz polynomial method we find some nonzero polynomial $p \in \mathbb{R}[x, y]$, of degree $\leq \sqrt{r}$, whose zero set Z decomposes $\mathbb{R}^2$ in $\simeq r$ cells, each with $\lesssim m/r$ points of P.

Let P_0 be the set of points of P which lie in Z, Γ_0 the set of curves of Γ which lie in Z, $C_1, ..., C_s$ the connected components of $\mathbb{R}^2 \setminus Z$, P_i the set of points of P which lie in C_i, and Γ_i the set of curves in Γ intersecting the interior of C_i, $\forall i = 1, ..., s$.

Since $I_{P,\Gamma} = I_{P_0,\Gamma_0} + I_{P_0,\Gamma\setminus\Gamma_0} + \sum_{i=1}^{s} I_{P_i,\Gamma_i}$, it suffices to bound each of these three quantities. We bound the last two as in the proof of [KMS11, Theorem 4.1], while for I_{P_0,Γ_0} we observe that $|\Gamma_0| \leq 2 \deg p + 1 = 2\sqrt{r} + 1$, which implies that $I_{P_0,\Gamma_0} \lesssim_b |P_0| + |\Gamma_0|^2 \lesssim_b m + \sqrt{r}^2 \sim_b m + r \lesssim_b m$.

We just explain why there exist at most $2 \deg p + 1 = 2\sqrt{r} + 1$ curves in Γ_0:

Suppose that there exist $2\sqrt{r}+2$ curves in Γ_0. Thanks to the type of parametrisation of the curves in Γ, these curves are unbounded and path-connected, while each two intersect at most b times. Therefore there exists some $\epsilon > 0$, some $x \in \mathbb{R}^2$ and some direction $\omega \in \mathbb{R}^2$, such that any line with direction perpendicular to ω, passing through a point of the interval $(x - \epsilon \cdot \omega, x + \epsilon \cdot \omega)$, intersects at least half of the curves of Γ_0, i.e. $\geq \sqrt{r}+1$ curves of Γ_0, at distinct points. Thus, the polynomial p vanishes on each of these lines, so it vanishes on a rectangle of $\mathbb{R}^2$. Therefore, p vanishes everywhere on $\mathbb{R}^2$, which is a contradiction.

$\square$

It obviously follows, by projecting $\mathbb{R}^3$ on a generic plane, that:

Lemma 4.1.11. *Let b be a positive constant. Also, let P be a finite set of points in $\mathbb{R}^3$ and Γ a finite family of curves in $\mathbb{R}^3$, such that each $\gamma \in \Gamma$ is parametrised by $t \to \big(p_1^\gamma(t), p_2^\gamma(t), p_3^\gamma(t)\big)$ for $t \in \mathbb{R}$, where the $p_i^\gamma \in \mathbb{R}[x,y,z]$, for $i = 1,2,3$, are polynomials not simultaneously constant, of degrees at most b. Then,*

(i) $I_{P,\Gamma} \lesssim_b |P|^{b/(2b-1)}|\Gamma|^{(2b-2)/(2b-1)} + |P| + |\Gamma|$, and

(ii) if there exist S points in P such that each lies in at least k curves of Γ, then $S \lesssim_b |\Gamma|^2/k^{(2b-1)/(b-1)} + |\Gamma|/k$.

Now, extending the proof of [GK08, Corollary 2.5] to a more general setting (see [GK08] or [CLO05, Chapter 3] for the definition and properties of the resultant $Res(f,g)$ of two polynomials f and g), we deduce the following:

Lemma 4.1.12. *Suppose that f, g are non-constant polynomials in $\mathbb{C}[x,y,z]$ which do not have a common factor. Then, the number of irreducible complex algebraic curves in $\mathbb{C}^3$ which are simultaneously contained in the zero set of f and the zero set of g is $\leq \deg f \cdot \deg g$.*

Proof. Let Γ be the family of irreducible complex algebraic curves in $\mathbb{C}^3$. Suppose that there exist $\deg f \cdot \deg g + 1$ curves in Γ, simultaneously contained in the zero set of f and the zero set of g. A generic complex plane intersects a complex algebraic curve in $\mathbb{C}^3$ at least once and finitely many times, while each two curves in Γ intersect in finitely many points of $\mathbb{C}^3$. Therefore, we can change the coordinates, so that f and g have positive degree in x, and also so that there exists some point $p = (p_1, p_2, p_3) \in \mathbb{C}^3$ and some $\epsilon > 0$, such that any plane in the family $\mathcal{A} := \big\{$planes in $\mathbb{C}^3$, perpendicular to $(0,0,1)$ and passing through a point of the interval $\big(p - \epsilon \cdot (0,0,1), p + \epsilon \cdot (0,0,1)\big)\big\}$ is transverse to all the $\deg f \cdot \deg g + 1$ curves, intersecting them at distinct points. Thus, each such plane contains at least $\deg f \cdot \deg g + 1$ distinct points of $\mathbb{C}^3$ where both f and g vanish.

Therefore, if $\Pi \in \mathcal{A}$, then $f_{|\Pi}$, $g_{|\Pi}$ are two polynomials in $\mathbb{C}[x,y]$, vanishing at $\geq \deg f \cdot \deg g + 1 \geq \deg f_{|\Pi} \cdot \deg g_{|\Pi} + 1$ distinct points of $\mathbb{C}^2$. And at the same time, there are at most $\deg f$, i.e. finitely many, planes Π in $\mathcal{A}$, such that $f_{|\Pi}$ does not have positive degree in x, and at most $\deg g$, i.e. finitely many, planes Π in $\mathcal{A}$, such that $g_{|\Pi}$ does not have positive degree in x.

Thus, there exists an open interval $I \subset \big(p - \epsilon \cdot (0,0,1), p + \epsilon \cdot (0,0,1)\big)$, such that if Π is a plane in the family $\mathcal{A}' := \big\{$planes in $\mathbb{C}^3$, perpendicular to $(0,0,1)$ and passing through a point of the interval $I\big\}$, then: $f_{|\Pi}$, $g_{|\Pi}$ are two polynomials in $\mathbb{C}[x,y]$, of positive degree in x, which are vanishing at $\geq \deg f_{|\Pi} \cdot \deg g_{|\Pi} + 1$ distinct points of $\mathbb{C}^2$. Thus, from Bézout's theorem, $f_{|\Pi}$ and $g_{|\Pi}$ have a common factor when seen as polynomials in $\mathbb{C}[y][x]$, and thus the polynomial $Res(f_{|\Pi}, g_{|\Pi}; x)$ (which is an element of $\mathbb{C}[y]$) is identically zero. However, $Res(f_{|\Pi}, g_{|\Pi}; x) \equiv Res(f,g;x)_{|\Pi}$.

As a result, we have that, $\forall\, \Pi \in \mathcal{A}'$, $Res(f,g;x)_{|\Pi} \equiv 0$; this means that the polynomial $Res(f,g;x) \in \mathbb{C}[y,z]$ vanishes $\forall (y,z) \in \mathbb{C}^2$, such that $y \in \mathbb{C}$ and $z \in J$, for some subset J of $\mathbb{C}$ of the form $(p_3 + a_1, p_3 + a_2)$, where $a_1, a_2 \in \mathbb{R}$. In other words, the polynomial $Res(f,g;x) \in \mathbb{C}[y,z]$ vanishes on a rectangle of $\mathbb{C}^2$. Therefore, it vanishes identically. And $Res(f,g;x) \equiv 0$ means that f and g have a common factor (since they both have positive degree when viewed as polynomials in x). We are thus led to a contradiction, which means that the exist $\leq \deg f \cdot \deg g$ curves of Γ simultaneously contained in the zero set of f and the zero set of g.

$\square$

Corollary 4.1.13. *Let f and g be non-constant polynomials in $\mathbb{R}[x,y,z]$. Suppose that f and g do not have a common factor. Let Γ be:*

(i) the family of irreducible real algebraic curves in $\mathbb{R}^3$, or

(ii) the family of real curves in $\mathbb{R}^3$, such that each $\gamma \in \Gamma$ is parametrised by $t \to \big(p_1^\gamma(t), p_2^\gamma(t), p_3^\gamma(t)\big)$ for $t \in \mathbb{R}$, where the $p_i^\gamma \in \mathbb{R}[x,y,z]$, for $i = 1,2,3$, are not simultaneously constant polynomials.

Then, the number of curves in Γ which are simultaneously contained in the zero set of f and the zero set of g is $\leq \deg f \cdot \deg g$.

Proof. Suppose that a curve $\gamma \in \Gamma$ lies in both the zero set of f and the zero set of g. We look at f and g as polynomials in $\mathbb{C}[x,y,z]$, and viewed as such we denote them by $f_{\mathbb{C}}$, $g_{\mathbb{C}}$, respectively. Also, for all $\gamma \in \Gamma$ we denote by $\gamma_{\mathbb{C}}$ the smallest irreducible complex algebraic curve containing γ (we have mentioned earlier that, in the case (ii), the smallest complex algebraic curve containing γ is, indeed, irreducible). It is easy to see that, if $\gamma^{(1)}$, $\gamma^{(2)} \in \Gamma$ with $\gamma^{(1)} \not\equiv \gamma^{(2)}$, then $\gamma_{\mathbb{C}}^{(1)} \not\equiv \gamma_{\mathbb{C}}^{(2)}$. Now, since the polynomials f, $g \in \mathbb{R}[x,y,z]$ do not have a common factor in $\mathbb{R}[x,y,z]$, the polynomials $f_{\mathbb{C}}$, $g_{\mathbb{C}} \in \mathbb{C}[x,y,z]$ do not have a common factor in $\mathbb{C}[x,y,z]$ (because if $h \in \mathbb{C}[x,y,z]$ is a common factor of $f_{\mathbb{C}}$, $g_{\mathbb{C}}$, which are polynomials with real coefficients, then $\bar{h} \in \mathbb{C}[x,y,z]$ is also a common factor of $f_{\mathbb{C}}$, $g_{\mathbb{C}}$, therefore $h\bar{h} \in \mathbb{R}[x,y,z]$ is a common factor of the polynomials f, $g \in \mathbb{R}[x,y,z]$). And we know that, as it contains γ, the irreducible complex algebraic curve $\gamma_{\mathbb{C}}$ intersects the zero set of $f_{\mathbb{C}}$ (which contains the zero set of f) infinitely many times; thus, by Bézout's theorem, it is contained in the zero set of $f_{\mathbb{C}}$. Similarly, $\gamma_{\mathbb{C}}$ is contained in the zero set of $g_{\mathbb{C}}$.

Thus, if $> \deg f \cdot \deg g$ curves of Γ lie simultaneously in the zero set of f and the zero set of g, then there exist $> \deg f \cdot \deg g$ irreducible complex

algebraic curves in $\mathbb{C}^3$, lying in both the zero set of $f_{\mathbb{C}}$ and the zero set of $g_{\mathbb{C}}$, where $f_{\mathbb{C}}$, $g_{\mathbb{C}} \in \mathbb{C}[x,y,z]$ do not have a common factor in $\mathbb{C}[x,y,z]$. By Lemma 4.1.10 though, this is a contradiction. Therefore, the number of curves in Γ which are simultaneously contained in the zero set of f and the zero set of g is $\leq \deg f \cdot \deg g$.

□

Corollary 4.1.14. *The zero set of a polynomial $p \in \mathbb{R}[x,y,z]$ contains at most $(\deg p)^2$ critical irreducible real algebraic curves of $\mathbb{R}^3$, and at most $(\deg p)^2$ critical curves parametrised by $t \to \big(p_1(t), p_2(t), p_3(t)\big)$ for $t \in \mathbb{R}$, where the $p_i \in \mathbb{R}[x,y,z]$, for $i = 1,2,3$, are polynomials not simultaneously constant.*

Proof. We can assume that p is square-free, as eliminating the squares of a polynomial does not inflict any change on its zero set. Thus, p and ∇p do not have a common factor, and the result follows by Corollary 4.1.13.

□

Finally, it is known (see [BPR06, Chapter 5]) that each real semi-algebraic set is the finite disjoint union of path-connected components. We observe the following:

Lemma 4.1.15. *A real algebraic curve in $\mathbb{R}^n$ is the finite disjoint union of $\lesssim_{b,n} 1$ path-connected components.*

Proof. This is obvious by a closer study of the algorithm in [BPR06, Chapter 5] that constitutes the proof of the fact that every real semi-algebraic set is the finite disjoint union of path-connected components.

□

4.2. **The general result.** We are now ready to formulate the following extension of Theorem 1.1:

Theorem 4.2.1. *Let b be a positive constant and Γ a finite collection of curves in $\mathbb{R}^3$, such that either*

(a) each $\gamma \in \Gamma$ is an irreducible real algebraic curve of degree at most b, or
(b) each $\gamma \in \Gamma$ is parametrised by $t \to \big(p_1^\gamma(t), p_2^\gamma(t), p_3^\gamma(t)\big)$ for $t \in \mathbb{R}$, where the $p_i^\gamma \in \mathbb{R}[x,y,z]$, for $i = 1,2,3$, are polynomials not simultaneously constant, of degrees at most b.

Then
$$\sum_{x \in J} N(x)^{1/2} \leq c \cdot |\Gamma|^{3/2},$$
where c_b is a constant depending only on b.

The proof of Theorem 4.2.1 is completely analogous to the proof of Theorem 1.1. Indeed, if γ is a real algebraic curve of degree at most b and $x \in \gamma$ is not an isolated point of γ, then γ crosses itself at x at most b times, while there exists exactly one tangent line to γ at x if γ does not cross itself at

x; thus there exist at least 1 and at most b tangent lines to γ at x. The same holds for real curves parametrised by not simultaneously constant real polynomials of degrees at most b. So, if x is a joint for a finite collection Γ of curves that satisfies (a) or (b), then each curve in Γ contributes at most b vectors to T_x^Γ, while, if x is a joint of multiplicity N for Γ, such that at most k curves of Γ, of which x is not an isolated point, are passing through x, then $N \leq (bk)^3$. Therefore, the following lemmas hold, whose statements and proofs are analogous to those of Lemmas 3.1 and 3.2:

Lemma 4.2.2. *Let x be a joint of multiplicity N for a collection Γ of curves in $\mathbb{R}^3$, such that either*

(a) each $\gamma \in \Gamma$ is a real algebraic curve of degree at most b, or
(b) each $\gamma \in \Gamma$ is parametrised by $t \to \big(p_1^\gamma(t), p_2^\gamma(t), p_3^\gamma(t)\big)$ for $t \in \mathbb{R}$, where the $p_i^\gamma \in \mathbb{R}[x, y, z]$, for $i = 1, 2, 3$, are polynomials not simultaneously constant, of degrees at most b.

Suppose that x lies in $\leq 2k$ of the curves in Γ of which it is not an isolated point. If, in addition, x is a joint of multiplicity $\leq N/2$ for a subcollection Γ' of Γ, or if it is not a joint at all for the subcollection Γ', then there exist $\geq \frac{N}{1000b^3 \cdot k^2}$ curves of $\Gamma \setminus \Gamma'$, of which x is not an isolated point, passing through x.

Lemma 4.2.3. *Let x be a joint of multiplicity N for a collection Γ of curves in $\mathbb{R}^3$, such that either*

(a) each $\gamma \in \Gamma$ is a real algebraic curve of degree at most b, or
(b) each $\gamma \in \Gamma$ is parametrised by $t \to \big(p_1^\gamma(t), p_2^\gamma(t), p_3^\gamma(t)\big)$ for $t \in \mathbb{R}$, where the $p_i^\gamma \in \mathbb{R}[x, y, z]$, for $i = 1, 2, 3$, are polynomials not simultaneously constant, of degrees at most b.

Suppose that x lies in $\leq 2k$ of the curves in Γ of which x is not an isolated point. Then, for every plane containing x, there exist $\geq \frac{N}{1000b^3 \cdot k^2}$ vectors of T_x^Γ which are not parallel to the plane.

In addition, as in the case of lines, we define

$J_N := \{x \in J : N \leq N(x) < 2N\}$, for all $N \in \mathbb{N}$, and

$J_N^k := \{x \in J_N$: x intersects at least k and less than $2k$ curves in Γ of which x is not an isolated point$\}$, for all $N, k \in \mathbb{N}$.

Then, Theorem 4.2.1 easily follows from Proposition 4.2.4, the statement and a sketch of the proof of which we now present:

Proposition 4.2.4. *Let $b \in \mathbb{N}$ and Γ a finite collection of curves in $\mathbb{R}^3$, such that either*

(a) each $\gamma \in \Gamma$ is a real algebraic curve of degree at most b, or
(b) each $\gamma \in \Gamma$ is parametrised by $t \to \big(p_1^\gamma(t), p_2^\gamma(t), p_3^\gamma(t)\big)$ for $t \in \mathbb{R}$, where the $p_i^\gamma \in \mathbb{R}[x, y, z]$, for $i = 1, 2, 3$, are polynomials not simultaneously constant, of degrees at most b.

Then

$$|J_N^k| \cdot N^{1/2} \leq c_b \cdot \left(\frac{|\Gamma|^{3/2}}{k^{1/(2D_b-2)}} + \frac{|\Gamma|}{k} \cdot N^{1/2} \right),$$

where D_b and c_b are constants depending only on b (and, in particular, $D_b \geq b+2$ in the case (a), while $D_b = b+2$ for $b \geq 2$ and $D_b = b+1$ for $b = 1$ in the case (b)).

Proof. Keeping in mind that a curve $\gamma \in \Gamma$ crosses itself at a point x at most b times, and therefore the number of tangent lines to γ at x is at most b, the proof is completely analogous to that of Proposition 1.2. The main differences lie at the beginning and the cellular case, we thus go on to point them out.

By Lemmas 4.1.7, 4.1.9 and 4.1.11, there exists an integer D_b, depending only on b, such that $D_b \geq b$ in the case (a), while $D_b = b+2$ for $b \geq 2$ and $D_b = b+1$ for $b = 1$ in the case (b), with the following properties:

(i) For all $\gamma \in \Gamma'$, where Γ' is a finite collection of curves in $\mathbb{R}^3$ that satisfies (a) or (b), γ crosses itself at most D_b times.

(ii) There exists at most 1 curve in Γ passing through any fixed $D_b - 1$ points in the case (b) for $b \geq 2$, as well as in the case (a), while there exists at most 1 curve in Γ passing through any fixed D_b points in the case (b) for $b = 1$.

(iii) For any finite collection Γ' of curves in $\mathbb{R}^3$ that satisfies (a) or (b), $|J_N^k| \lesssim_b |\Gamma'|^2/k^{(2D_b-1)/(D_b-1)} + |\Gamma'|/k$.

For these specific D_b in each case, the proof of the proposition will be achieved by induction on the cardinality of $|\Gamma|$. Indeed, let $M \in \mathbb{N}$. For c_b an explicit constant $\geq D_b$, which depends only on b and will be specified later:

- For any family of curves in $\mathbb{R}^3$ that satisfies (a) or (b) and consists of 1 curve,

$$|J_N^k| \cdot N^{1/2} \leq c_b \cdot \left(\frac{1^{3/2}}{k^{1/(2D_b-2)}} + \frac{1}{k} \cdot N^{1/2} \right), \ \forall \ N, k \in \mathbb{N}$$

(this is obvious, in fact, for any $c_b \geq D_b$, as in this case $|J_N| = |J_N^1| \leq D_b$ $\forall \ N \in \mathbb{N}$, since a curve in a family satisfying (a) or (b) crosses itself at most D_b times).

- We assume that

$$|J_N^k| \cdot N^{1/2} \leq c_b \cdot \left(\frac{|\Gamma'|^{3/2}}{k^{1/(2D_b-2)}} + \frac{|\Gamma'|}{k} \cdot N^{1/2} \right), \ \forall \ N, k \in \mathbb{N},$$

for any collection Γ' of curves in $\mathbb{R}^3$ that satisfies (a) or (b), with $|\Gamma'| \lneq M$.

- We will now prove that

$$|J_N^k| \cdot N^{1/2} \leq c_b \cdot \left(\frac{|\Gamma|^{3/2}}{k^{1/(2D_b-2)}} + \frac{|\Gamma|}{k} \cdot N^{1/2} \right), \tag{4}$$

$\forall\ N, k \in \mathbb{N}$, for any collection Γ of curves in $\mathbb{R}^3$ that satisfies (a) or (b), with $|\Gamma| = M$.

Indeed, let Γ be a collection of curves in $\mathbb{R}^3$ that satisfies (a) or (b), with $|\Gamma| = M$. Fix N and k in $\mathbb{N}$. As in the case of lines, let $\mathfrak{G} := J_N^k$ and $S := |J_N^k|$ for this collection Γ, while Γ' denotes the set of curves in Γ each of which contains $\geq \frac{1}{100} Sk|\Gamma|^{-1}$ points of $\mathfrak{G}$ which are not isolated points of the curve.

Now, we know that $S \cdot N^{1/2} \leq c_{0,b} \cdot (|\Gamma'|^2/k^{(2D_b-1)/(D_b-1)} + |\Gamma'|/k)$ for some constant $c_{0,b}$ depending only on b. Thus:

If $\frac{S}{2} \leq c_{0,b} \cdot |\Gamma| k^{-1}$, then $S \cdot N^{1/2} \leq 2c_{0,b} \cdot \frac{|\Gamma|}{k} \cdot N^{1/2}$ (where $2c_{0,b}$ is independent of Γ, N and k).

Otherwise, $\frac{S}{2} < c_{0,b} \cdot |\Gamma|^2/k^{(2D_b-1)/(D_b-1)}$, so $S < 2c_{0,b} \cdot |\Gamma|^2 k^{-(2D_b-1)/(D_b-1)}$.

Therefore, the quantity $d := A|\Gamma|^2 S^{-1} k^{-(2D_b-1)/(D_b-1)}$ is equal to at least 1 whenever $A \geq 2c_{0,b}$; we thus choose A to be large enough for this to hold, and we will specify its value later. Now, using the Guth - Katz polynomial method, we find a polynomial p of degree $\leq d$, whose zero set Z:

(i) decomposes $\mathbb{R}^3$ in $\lesssim_b d^3$ cells, each of which contains $\lesssim_b Sd^{-3}$ points of $\mathfrak{G}$, and

(ii) contains 6 distinct generic planes, each of which contains a face of a fixed cube Q in $\mathbb{R}^3$, such that the interior of Q contains $\mathfrak{G}$ (and each of the planes is generic in the sense that the plane in $\mathbb{C}^3$ containing it intersects the smallest complex algebraic curve in $\mathbb{C}^3$ containing γ, for all $\gamma \in \Gamma$);

to achieve this, we firstly fix a cube Q in $\mathbb{R}^3$, with the property that its interior contains $\mathfrak{G}$ and the planes containing its faces are generic in the above sense. Then, we multiply the polynomials we end up with at each step of the Guth - Katz polynomial method with the same (appropriate) 6 linear polynomials, the zero set of each of which is a plane containing a different face of the cube, and stop the application of the method when we finally get a polynomial of degree at most d, whose zero set decomposes $\mathbb{R}^3$ in $\lesssim_b d^3$ cells (the cells now are the intersections of the cube Q with the cells that arise from the application of the Guth - Katz polynomial method).

If there are $\geq 10^{-8}S$ points of $\mathfrak{G}$ in the union of the interiors of the cells, we are in the cellular case. Otherwise, we are in the algebraic case.

Cellular case: There are $\gtrsim S$ points of $\mathfrak{G}$ in the union of the interiors of the cells. However, we also know that there exist $\lesssim_b d^3$ cells in total, each with $\lesssim_b Sd^{-3}$ points of $\mathfrak{G}$. Therefore, there exist $\gtrsim_b d^3$ cells, with $\gtrsim_b Sd^{-3}$ points of $\mathfrak{G}$ in the interior of each. We call the cells with this property "full cells". Now:

• If the interior of some full cell contains $< k^{1/(D_b-1)}$ points of $\mathfrak{G}$, then $Sd^{-3} \lesssim_b k^{1/(D_b-1)}$, and since $N \lesssim b^3k^3 \lesssim_b k^3$, we have that $S \cdot N^{1/2} \lesssim_b |\Gamma|^{3/2}/k^{1/(2D_b-2)}$.

• If the interior of each full cell contains $\geq k^{1/(D_b-1)}$ points of $\mathfrak{G}$, then we will be led to a contradiction by choosing A sufficiently large. Indeed:

Consider a full cell and let $\mathfrak{G}_{cell}$ be the set of points of $\mathfrak{G}$ lying in the interior of the cell, S_{cell} the cardinality of $\mathfrak{G}_{cell}$ and $\Gamma_{cell} := \{\gamma \in \Gamma : \exists\, x \in \gamma \cap \mathfrak{G}_{cell},$ such that x is not an isolated point of $\gamma\}$.

Let $\mathfrak{G}'_{cell}$ be a subset of $\mathfrak{G}_{cell}$ of cardinality $k^{1/(D_b-1)}$. Since each point of $\mathfrak{G}_{cell}$ has at least k curves of Γ_{cell} passing through it, there exist at least $k^{D_b/(D_b-1)}$ incidences between Γ_{cell} and $\mathfrak{G}'_{cell}$. On the other hand, the curves in Γ_{cell} containing at most D_b-2 points of $\mathfrak{G}'_{cell}$ contribute at most $(D_b-2)\cdot|\Gamma_{cell}|$ incidences with $\mathfrak{G}'_{cell}$, while there exist at most $\binom{|\mathfrak{G}'_{cell}|}{D_b-1}$ curves in Γ_{cell} such that each contains at least D_b-1 points of $\mathfrak{G}'_{cell}$, since there exists at most 1 curve of Γ passing through any fixed D_b-1 points. Thus,

$$k^{D_b/(D_b-1)} \leq I_{\mathfrak{G}'_{cell},\Gamma_{cell}} \leq (D_b-2)\cdot|\Gamma_{cell}| + \binom{k^{1/(D_b-1)}}{D_b-1}\cdot k^{1/(D_b-1)} \leq$$

$$\leq (D_b-2)\cdot|\Gamma_{cell}| + \frac{1}{(D_b-1)!}\cdot k^{D_b/(D_b-1)}, \text{ so}$$

$$|\Gamma_{cell}| \gtrsim_b k^{D_b/(D_b-1)},$$

and thus

$$|\Gamma_{cell}|^2/k^{(2D_b-1)/(D_b-1)} \gtrsim_b |\Gamma_{cell}|/k.$$

Note that in the case (b) for $b=1$ we can get the same result for $D_b=b+1$, by applying the same idea as in the case of lines.

However, D_b was chosen in such a way that

$$S_{cell} \lesssim_b |\Gamma_{cell}|^2/k^{(2D_b-1)/(D_b-1)} + |\Gamma_{cell}|/k$$

(note that each of the points in $\mathfrak{G}_{cell}$ has at least k curves of Γ_{cell} passing through it, of each of which it is not an isolated point).

Therefore, $S_{cell} \lesssim_b |\Gamma_{cell}|^2/k^{(2D_b-1)/(D_b-1)}$, so, since we are working in a full cell, $Sd^{-3} \lesssim_b |\Gamma_{cell}|^2/k^{(2D_b-1)/(D_b-1)}$, and rearranging we see that

$$|\Gamma_{cell}| \gtrsim_b S^{1/2}d^{-3/2}k^{(2D_b-1)/(2D_b-2)}.$$

Furthermore, let Γ_Z be the set of curves of Γ which are lying in Z and Γ'_{cell} the set of curves in Γ_{cell} such that, if $\gamma \in \Gamma'_{cell}$, there does not exist any point x in the intersection of γ with the boundary of the cell, with the property that the induced topology from $\mathbb{R}^3$ to the intersection of γ with the closure of the cell contains some open neighbourhood of x. Obviously, $\Gamma_{cell} \subset \Gamma \setminus \Gamma_Z$, and in the case (b), $\Gamma'_{cell} = \emptyset$ (by the construction of p and the fact that each $\gamma \in \Gamma$ is unbounded and path-connected in this case). Finally, let $I_{cell} := |\Gamma_{cell} \setminus \Gamma'_{cell}|$; note that the number of incidences between the boundary of the cell and the curves in $\Gamma_{cell} \setminus \Gamma'_{cell}$ is $\geq I_{cell}$.

Let us now work in case (b) only: in this case, each $\gamma \in \Gamma$ is path-connected and unbounded, while $\mathfrak{G}$ is contained in a cube Q, whose boundary is contained in Z. Therefore, each of the curves of Γ_{cell} intersects the boundary of the cell at at least one point, with the property that the induced topology from $\mathbb{R}^3$ to the intersection of the curve with the closure of the cell contains an open neighbourhood of x; therefore, $I_{cell} = |\Gamma_{cell}| \gtrsim_b S^{1/2}d^{-3/2}k^{(2D_b-1)/(2D_b-2)}$. Also, the union of the boundaries of all the cells is the zero set Z of p, and if x is a point of Z which belongs to a curve of Γ intersecting the interior of a cell, such that the induced topology from $\mathbb{R}^3$ to the intersection of the curve with the closure of the cell contains an open neighbourhood of x, then there exist at most $2b-1$ other cells whose interior is also intersected by the curve and whose boundary contains x, such that the induced topology from $\mathbb{R}^3$ to the intersection of the curve with the closure of each of these cells contains some open neighbourhood of x. So, if I is the number of incidences between Z and $\Gamma \setminus \Gamma_Z$, and $\mathcal{C}$ is the set of all the full cells (which, in our case, has cardinality $\gtrsim_b d^3$), then,

$$I \geq \frac{1}{2b} \cdot \sum_{cell \in \mathcal{C}} I_{cell} \gtrsim_b (S^{1/2}d^{-3/2}k^{(2D_b-1)/(2D_b-2)}) \cdot d^3 = S^{1/2}d^{3/2}k^{(2D_b-1)/(2D_b-2)}.$$

while each $\gamma \in \Gamma$ intersects Z at most $b \cdot \deg p$ times.

In the case (a) though, it is not necessarily true that $\Gamma'_{cell} = \emptyset$, so our approach has to be a little different. Indeed, if $\gamma \in \Gamma'_{cell}$, we consider the smallest complex algebraic curve $\gamma_{\mathbb{C}}$ in $\mathbb{C}^3$ which contains γ. In addition, let $p_{\mathbb{C}}$ be the polynomial p seen in $\mathbb{C}[x,y,z]$ and $Z_{\mathbb{C}}$ the zero set of $p_{\mathbb{C}}$ in $\mathbb{C}^3$. The polynomial p was constructed in such a way that $\gamma_{\mathbb{C}}$ intersects each of 6 complex planes, each of which contains one of the real planes in Z that contain the faces of the cube Q; consequently $\gamma_{\mathbb{C}}$ intersects $Z_{\mathbb{C}}$ at least once. So, if $\Gamma_{\mathbb{C}} = \{\gamma_{\mathbb{C}} : \gamma \in \Gamma_{cell}$, for all cells in $\mathcal{C}\}$ and $I_{\mathbb{C}}$ denotes the number of incidences between $\Gamma_{\mathbb{C}}$ and $Z_{\mathbb{C}}$, it follows that in the case (a):

$$I_{\mathbb{C}} \geq \frac{1}{2b} \cdot \sum_{cell \in \mathcal{C}} I_{cell} \text{ and } I_{\mathbb{C}} \geq |\Gamma_{\mathbb{C}}| = |\Gamma| \geq \Big|\bigcup_{cell \in \mathcal{C}} \Gamma'_{cell}\Big|,$$

thus

$$I_{\mathbb{C}} \geq \frac{1}{2} \cdot \left(\frac{1}{2b} \cdot \sum_{cell \in \mathcal{C}} I_{cell} + \Big|\bigcup_{cell \in \mathcal{C}} \Gamma'_{cell}\Big|\right) \gtrsim_b$$

$$\gtrsim_b \sum_{cell \in \mathcal{C}} I_{cell} + \Big|\bigcup_{cell \in \mathcal{C}} \Gamma'_{cell}\Big|.$$

However, γ is the disjoint union of $\leq R_b$ path-connected components, for some constant R_b depending only on b (by Lemma 4.1.15). Therefore, the above quantity is

$$\gtrsim_b \sum_{cell \in \mathcal{C}} I_{cell} + \sum_{cell \in \mathcal{C}} |\Gamma'_{cell}| \gtrsim_b$$

$$\gtrsim_b \sum_{cell \in \mathcal{C}} (S^{1/2}d^{-3/2}k^{(2D_b-1)/(2D_b-2)} - |\Gamma'_{cell}|) + \sum_{cell \in \mathcal{C}} |\Gamma'_{cell}| \simeq_b$$

$$\simeq_b (S^{1/2}d^{-3/2}k^{(2D_b-1)/(2D_b-2)}) \cdot d^3 \simeq_b S^{1/2}d^{3/2}k^{(2D_b-1)/(2D_b-2)},$$

while each $\gamma_{\mathbb{C}} \in \Gamma_{\mathbb{C}}$ is a complex algebraic curve of degree at most b which does not lie in $Z_{\mathbb{C}}$, and thus intersects $Z_{\mathbb{C}}$ at most $b \cdot \deg p$ times.

So, in both (a) and (b),

$$S^{1/2}d^{3/2}k^{(2D_b-1)/(2D_b-2)} \lesssim_b |\Gamma| \cdot d,$$

which in turn gives: $A \lesssim_b 1$. In other words, there exists some constant C_b, depending only on b, such that $A \leq C_b$. By fixing A to be a number larger than C_b (and of course $\geq 2c_{0,b}$, so that $d > 1$), we have a contradiction.

Therefore, in the cellular case there exists some constant $c_{1,b}$, depending only on b, such that

$$S \cdot N^{1/2} \leq c_{1,b} \cdot \frac{|\Gamma|^{3/2}}{k^{1/(2D_b-2)}}.$$

The rest of the proof is just an adaptation of the proof of Proposition 1.2, using Corollary 4.1.14 and Lemmas 4.2.2 and 4.2.3.

□

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